Teaching Fractions

What’s the big deal?

Dr. Cathy Bruce
Supported by Shelley Yearley

November 2014
Agenda

- current research on what makes fractions challenging for students
- what foundational fractions concepts need more attention
- which representations are helpful to students
- key strategies for supporting educators who are teaching fractions
- Interaction through the chat window, polls and an experiment with a whiteboard
1. Why is fractions learning so challenging?

What the research says...
What we know from research

- Fractions is treated as a discrete topic

“fractions unit” tends to be taught in isolation from other math content

Even though fractions is inherent in much of the mathematics … “I’m teaching measurement now, I am not doing fractions”

…. 

Persistence with whole number counting and contexts in school math does not help…
Continuous vs Whole number thinking

- We rote count 1, 2, 3, 4,….
- We don’t count one, one and one-half, two, two and one-half… etc. Unless we are *really stalling*
- We focus on whole numbers all through primary (and junior) grades, and then switch to the continuous number system hoping students will transfer the concepts

- Two problems:
  a. continuous and whole number thinking are very different;
  b. we have lost the child’s intuitive understandings of rational numbers
Understanding Density


Only 24% [of 4\textsuperscript{th} and 5\textsuperscript{th} graders] knew that there was an infinite amount of numbers between any two numbers.

43% claimed that there are no numbers between _ and _.
Think of a metaphor...

- What might you use as a metaphor to describe the density of the number line – say between 1 and 2?
More challenges...

- We conflate the meanings of fractions and their representations

Set model

Part-whole fractions thinking (not part-part)

Could we think about this as part-part?

Problem 3
Multitude of representations in North American textbooks

Circular area models are overrepresented and pose difficulties (difficult to equipartition, precedes introduction of deep study the circle)
Students attempting to partition...\textbf{Partitioning Circles}

\begin{center}
\includegraphics[width=\textwidth]{partitioning_circles.png}
\end{center}

HMMMMM......
Multiple Meanings of Fractions

- Linear measure (a distance from 0 in either direction)
- Part-whole relationship (relative quantity, often area-based, but can also be set-based)
- Part-part relationship (ratios)
- Fraction as operator (multiplicative action on quantity)
- Fraction as quotient (division relationship)

A fraction is one quantity or amount
Ways we think about fractions

A fraction is a number, which can tell us about the relationship between two quantities. These two quantities provide information about the parts, the units we are considering and the whole. Determining the whole is important when working with fractions.

**Thinking about fractions as unit fractions**

This is ‘two-sixths’

but we could think about this as units of one-sixth

1 one-sixth, 2 one-sixths

or we could also think about it as “two times the unit created by partitioning the whole into six equal parts”

Understanding unit fractions helps students develop fractions number sense and supports work with fractions operations.

\( \frac{3}{6} \) can represent one or more of the following, depending on the context:

- a **linear measure** (read ‘two-sixths’).
  - The number is defined by its distance from 0.

- a **part-whole relationship** (read ‘two-sixths’) which is based upon a relative measure or a set:
  - two-dimensional measures, such as an area or region, where an equal part is an equal area.
  - three-dimensional measures, such as capacity or mass, where an equal part is an equal capacity or mass.
  - sets, such as a collection of objects, where an equal part is an item in the set and equal parts are not necessarily identical. Attributes (e.g., colour, size, shape) may or may not be considered. For example, colour is important for the first set but not in the second two. Parts can be organized in an array, randomly, or in composite sets.

- a **part-part relationship** (read ‘two to six’).
  - The parts are equivalent with respect to one attribute (not necessarily size). Ratio is frequently used for part to part relationships.

- a **quotient**
  - \( \frac{a}{b} \) = a divided by b

- an **operator**
  - \( \frac{a}{b} \) of a quantity (the whole)
  - The whole is assumed to be the quantity being multiplied by \( \frac{a}{b} \). The fraction acts as a transformer by either enlarging or shrinking the operand.

Linear measure example: 2 cm to 6 cm

Two-dimensional measure example:
2 equal areas shaded to 6 equal areas unshaded
Set example:
2 apples to 6 bananas

\( \frac{3}{6} = \frac{2}{6} \) or 2 partitioned into 6 equal parts

Two-dimensional measure example:
\( \frac{3}{4} \) of the area of the floor
Set example:
\( \frac{2}{4} \) of the people in the room

See the Fractions Digital Paper at www.edugains.ca.

Www.edugains.ca
Or go from:
tmerc.ca
(fractions tab)
Part-whole Relationships

Linear models

2D Area models

3D models (volume; surface area)
Part-whole Set Relationships

\[ \frac{2}{6} \]
Part-part Relationships

- 2 parts orange concentrate to 6 parts water
- 2:6

Put another part-part ratio example into the chat space now.
The Research is Clear
Mooseley’s 2005 research:

- demonstrated that students who were familiar with both the part-part and part-whole interpretations had a deeper understanding of rational numbers.

- highlighted the need to expand students’ fractions understanding beyond the typical meaning that is focused on in mathematics programs in North America—fractions as part-whole relationships.

Based upon research by Dr. Cathy Bruce, Trent University in partnership with the Curriculum and Assessment Policy Branch
Behr, Harel, Post & Lesh (1993) have state that

“...learning fractions is probably one of the most serious obstacles to the mathematical maturation of children.”
Leads to challenges in work and life...

- Calculation of dosage errors
- Measure twice, but still cut 3 times
- Estimations of time
- Conservation of liquid
- Baking
Fractions Research Team
2011-2014

Tara Flynn

Cathy Bruce

Sarah Bennett

Rich McPherson

Shelley Yearley

Curriculum and Assessment Policy Branch

DSBN
HWDSB
KPRDSB
OCDSB
SCDSB
SMCDSB
TLDSB
YCDSB

Social Sciences and Humanities Research Council of Canada
Conseil de recherches en sciences humaines du Canada
Collaborative Action Research

**CLASSROOM**

**Collaboration**
Throughout collaborative action research activity, teachers frequently meet to set goals, and to plan and engage in related interventions, data collection, data analysis and report writing. The involvement of researchers and knowledgeable others can range from full membership in the team to a supporting role (e.g., providing resources, assisting with data collection and analysis strategies).

**Identifying the Problem**
- articulating the teacher/learning issue for investigation
- gathering baseline data
- developing research question(s)

**Goal Setting and Planning**
- planning interventions that will improve teaching and learning
- consulting current research and accessing human/print resources
- setting timelines

**Implementing Plan of Action**
- enacting the plan in the classroom
- observing, co-teaching, supporting team members
- collecting evidence and reflecting

**Evaluating Effects**
- analyzing data
- report writing
- sharing reports

**Asking New Questions**
The findings of collaborative action research often reveal other problems and lead to new questions that provide a springboard for a second cycle of action research in new or refined areas of study.

**Data Collection and Analysis**
Context and Participants

- 8 school boards
  - Kawartha Pine Ridge
  - Trillium Lakelands
  - Ottawa Carleton
  - DSB Niagara
  - Simcoe County
  - Simcoe Muskoka Catholic
  - York Catholic
  - Hamilton Wentworth

- 75 classroom teachers

- 1000+ students (to date)

- Shortest interval 4 months; longest 8 months
Data Collection

■ STUDENTS
  ▪ video of student thinking
  ▪ written student responses
  ▪ clinical interviews
  ▪ Pre-post tests

■ TEACHERS
  ▪ video of discussion
  ▪ lessons and instructional sequences

Pre- and post-data initially collected using a tool validated by Ross and Bruce (2009), then revised, by Bruce, Flynn & Yearley, field-tested and validated again (design research process)
Overarching Research Question

What instructional strategies and tools support junior and intermediate grade students in learning fractions?
1. Show five-sixths in two ways.

2. \( \frac{3}{4} \) is the same as \( \frac{15}{20} \). How do you know?

3. The sum of \( \frac{1}{12} \) and \( \frac{7}{8} \) is closest to:
   a. 20  
   b. 8  
   c. \( \frac{1}{2} \)  
   d. 1
Year 1 Research Findings: Punctuated Instruction

Sample data from fractions research, where students worked with fractions throughout the year instead of as a stand alone "unit" of study.

Increase
No change
Decline

Bruce, Flynn, Yearley 2012
Year 3: Grade 4 PD Outcomes

Based upon research by Dr. Cathy Bruce, Trent University and one district school board.

All classes
Pre: 5.26
Post: 11.14
### Year 3: Pre-post Mean Differences

<table>
<thead>
<tr>
<th>Board</th>
<th>Grades</th>
<th>n</th>
<th>Pre-Post Mean Difference</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3-6</td>
<td>132</td>
<td>+29.81%</td>
<td>1.23</td>
</tr>
<tr>
<td>2</td>
<td>3-9</td>
<td>172</td>
<td>+25.91%</td>
<td>1.07</td>
</tr>
<tr>
<td>3</td>
<td>6-8</td>
<td>83</td>
<td>+10.33%</td>
<td>.43</td>
</tr>
</tbody>
</table>

Based upon research by Dr. Cathy Bruce, Trent University and Curriculum and Assessment Policy Branch and three DSBs.
Based upon research by Dr. Cathy Bruce, Trent University and Curriculum and Assessment Policy Branch and three DSBs
Fractions Learning Pathways

Please Note:
- Mixed, improper and proper fractions should be interspersed throughout fractions teaching and learning so that the students build flexibility with these early.
- "Models" include linear, area, volume, and set representations.
Time for some math!

Two-fifths of the marbles are blue and three-tenths of the marbles are yellow. Are there more blue or yellow marbles in the bag?
Does the number line help make sense of this situation?

- Using a linear model (Petit, p. 156)
- $\frac{2}{5}$ of the marbles are blue
- $\frac{3}{10}$ of the marbles are green
- Are there more blue or mint green marbles in the bag?

A comparing fractions situation
2. What foundational concepts need attention?

Unit fractions
Explicit Instruction of Unit Fractions

- Students should have the experience of composing and decomposing fractions using unit fractions:

  e.g., \( \frac{1}{4} \), \( \frac{1}{2} \)  
  The ‘4’ or the ‘2’ are not the whole, they are the unit we are counting.

  e.g., \( \frac{2}{5} \)  
  is ‘2 one-fifth units’

- Use of unit fractions allows students to make connections between the different constructs of numbers and “provides a catalyst for students’ transition from whole to rationale numbers” (Charalambous et al., 2010).
Let’s Count

**Fifths**

1 one-fifth units
2 one-fifths
3 one-fifths
4 one-fifths
5 one-fifths
6 one-fifths
7 one-fifths

**Sixths**

- Count aloud by sixths
- When you reach a whole, say BUZZ instead of the fraction quantity
Unit Thinking is Powerful

It allows students to connect their understanding of operations across number systems.

<table>
<thead>
<tr>
<th>Whole Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \begin{array}{c} 2 + 3 \ 2 \ 1 \text{ units} \ + 3 \ 1 \text{ units} \ 5 \ 1 \text{ units} \end{array} \ = 5 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Decimal Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \begin{array}{c} 0.2 + 0.3 \ 0.1 \text{ units} \ + 3 \ 0.1 \text{ units} \ 5 \ 0.1 \text{ units} \end{array} \ = 0.5 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{2}{7} + \frac{3}{7} ] [ 2 \ 1 \text{ units} ] [ 3 \ 1 \text{ units} ] [ \frac{5}{7} \text{ units} ] [ = \frac{5}{7} ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \begin{array}{c} 2a + 3a \ 2 \ a \text{ units} \ + 3 \ a \text{ units} \ 5 \ a \text{ units} \end{array} \ = 5a ]</td>
</tr>
</tbody>
</table>
Use Unit Fractions Reasoning

Which is greater: \( \frac{7}{8} \) or \( \frac{5}{8} \)?

How do you know?

Which is greater: \( \frac{3}{12} \) or \( \frac{3}{10} \)?

How do you know?
Parker Comparing Fractions

How do these fractions compare?

Record the inequality and use audio to explain why one fraction is greater than the other.

Bigger

Smaller
Implications

- Engage students in composing and decomposing fractions using unit fractions
- Have students name the units, including counting by unit fractions (1 one-fifth, 2 one-fifths, 3 one-fifths, …)
- Come back to unit fractions regularly – including for operations with fractions
3. Powerful representations
We have explored linear, area, and set models in detail.

Which representation is helpful in which situations?

Tad Watanabe, 2002 TCM article
Representations

The use of linear (and some area) visual representations, such as:

- number lines
- bars, strips or ribbons

enable students to think about partitioning and iterating as relative length or distance (or area) relationships, which is a particularly powerful way to make sense of fractional parts.

*Math teaching for learning: Developing Proficiency with Partitioning, Iterating, and Disembedding*, 2013

Based upon research by Dr. Cathy Bruce, Trent University and Curriculum and Assessment Policy Branch
Why Number Lines?

- To further develop understanding of fraction size (PROPORTIONAL REASONING)
- To see that the interval between two fractions can be further partitioned (DENSITY)
- To see that the same point on the number line represents an infinite number of equivalent fractions (EQUIVALENCY)
### Examining Representation Use

**Grade 6 Correct**
- Circle: 6
- Rectangle: 21
- Number line: 1
- Set: 5
- Symbols: 5
- Words: 1

**Grade 7 Correct**
- Circle: 3
- Rectangle: 6
- Number line: 2
- Set: 2
- Symbols: 2
- Words: 2

**Grade 6 Incorrect**
- Circle: 11
- Rectangle: 2
- Number line: 1
- Set: 1
- Symbols: 1
- Words: 1

**Grade 7 Incorrect**
- Circle: 1
- Rectangle: 1
- Number line: 1
- Set: 1
- Symbols: 5
- Words: 4
Walking to school; three-fifths of the way there

Comparing one-fourth of a glass of orange juice to two-sevenths of a glass of grape juice

Finding the ratio of red stones to white stones in the basket

Adding 4 one-fifths and 6 one-fifths

Multiplying one-sixth and one-third
Multiplying one-sixth and one-third

Multiplication can be understood as:
- Groups of (set)
- Repeated addition (linear)
- Shared space (area)

One-eighteenth of the area model is shaded green
4. Helping educators
Key Strategies

- Inquiry about fractions: Find a buddy, try tasks together and watch/listen to students
- Slow it down to speed it up: do one thing well – it will travel
- Be thoughtful about tasks and representations
- Use existing resources to support learning—digital paper, lessons, video studies

tmerc.ca & edugains sites
Q and A

- We have been monitoring your questions throughout the session.
- Please ask a key question in the chat space now, and we will try to address that, or at least open up a discussion and some additional thinking.