A Diversity of Ways to Support Active Learning of Mathematics.

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Mathematics and Statistics are plural words. They offer an inherent diversity of approaches, of mathematical tools and of mathematical processes that support problem solving. Our children bring rich and diverse abilities and interests connected to math into our classrooms. To respect the nature(s) of math, and our children, we need a continuing commitment to differentiated learning, differentiated instruction, and differentiated assessment. I want to highlight a few of pieces of research and classroom approaches to support this.

The current curriculum speaks to the use of multiple representations (say for the concept of function: numeric, graphic, symbolic, function machine). This diversity of representations applies much more broadly – in fact in every area of mathematics and statistics that I have experienced. Multiple representations offer multiple ways to access problems, and multiple ways to work with the selected concepts. Some students need a particular representation to enter the representational network of a concept. The strong students benefit from access to all of the representations and associated ways to problem solve – and their brain scans show more rapid switching among brain areas (and by inference among representations and strategies) than the 'average' students.

Cognitive blending offers a description of how humans combine approaches and multiple representations into an increasingly powerful network that becomes 'The Way We Think' [1]. In mathematics, this ubiquitous human process maps selected parts of several prior mathematical processes and representations together into a larger conceptual network that can now access multiple ways to solve new problems, making new links between parts of the prior learning that has now become blended [2].

Within blended conceptual development, we still retain access to some ways (perhaps multiple ways) the mathematics developed from the senses, and build ways we continue to use the senses for reasoning, and re-presentations (if we access them). These representations should include appropriate use of kinesthetic and visual/spatial senses, and classroom activities that (re)-connect to these, regularly. These are two of Gardner's Multiple Intelligences [3] that are identified in young children, and continue as metaphors for clusters of ways children approach many activities, including mathematical problem solving. For example, we can watch the gestures and hear metaphorical words that point to a child's active mental processes for problem solving and processing mathematics. Materials, including manipulatives and appropriate technology, which support different entry points are essential to supporting the diversity of students and diversity of ways mathematics can be effectively be learned and applied.

Spatial Reasoning as a key ability and a predictor of success in Science Technology, Engineering, Arts and Mathematics – the STEAM disciplines. In our current curriculum, many students emerge from high school with clear deficits in spatial and kinesthetic reasoning, something that has been identified as a barrier for retention and success in Engineering programs (particularly for women) [4]. Some research (with the late Margaret Sinclair) found that many elementary teachers felt unprepared for spatial reasoning and in some cases were very afraid to try this type of reasoning. They did want to develop this ability, if possible – and the good news is spatial reasoning is malleable – learnable – at all ages. I am currently developing spatial reasoning modules for first year engineering, first year science, and pre-service teachers at York, to try to address these gaps.

Children enter school engaged in spatial reasoning – but we often do not support regular development of these abilities. Spatial reasoning and kinesthetic reasoning don't disappear as people move towards "mathematical maturity" [5,6], except perhaps in the classroom and the books. Spatial reasoning and visual communication of this reasoning actually becomes more important as you move through rich undergraduate and graduate programs, particularly with interdisciplinary work within research teams from several disciplines – where the spatial/visual and kinesthetic approaches are more easily shared than many other approaches.

Here is a related classroom example. At the end of the geometry course which was rich in group work, the use of technology (GSP), and the use of manipulatives, I ask my classes of pre-service teachers, and in-service teachers, two questions: *Could you have learned the mathematics you did without these supports? Do you now see geometry everywhere?* With nuances, the answers are*: I would not have learned what I did without these supports*; and y*es I now see geometry (almost) everywhere!* Realizing that they learned well through these activities makes it more likely that these teachers will also look for ways to use these approaches to support student learning in their own classrooms.

Our recent research on tasks with manipulatives supporting experiences relating to change of volume and surface area gives some insights into how widely these approaches based on spatial reasoning transfer, with appropriate support. With structured investigations, many concepts we associate with upper level topics (precalculus) make the related mathematical concepts accessible to much younger students [7,8]. So a task originally developed for university students and teachers, to support the topic of optimization of volume, has been shifted through multiple levels of high school, and more recently brought to Grade 5 students. This experience confirms again that children are capable of richer work than the curriculum anticipates, and often more mathematics than we prepare teachers to support.

This diversity of activities can sound chaotic, and perhaps scary within our curricular models. There is no single trajectory that is best for all students, or for one student on different days! However, a variety of small experiments confirm that classrooms with rich, multidisciplinary problem solving activities end up including more mathematics than the curriculum, and with deep engaged understanding. Examples such as the Reggio Emelia programs (centered on the children's questions in early grades) and the Manchester Experiment from the 1930s (no formal math till grade 6) confirm that students can learn much more when we offer such engaged approaches, with good support for the teachers [9,10]. The first challenge then becomes to support the students going more places in math, and asking unexpected questions! The related challenge is to adequately support teachers who are asked to work in this ways.

Teachers' knowledge of the child's development of mathematics and the connections of the mathematics makes a difference to effective student engagement with the mathematics, particularly when students ask questions off the script. There is growing evidence both that teachers' knowledge of mathematics matters to the learning of their students, and that the current B.Ed. programs (and possibly future programs) do not support all teachers developing sufficient mastery of the relevant content knowledge and sufficient comfort to actively explore the mathematics and respond to the mathematical curiosity of their students [11,12].

There is a useful distinction related to the order and the depth of learning, which continues to inform both research and classroom practices: Skemp's Instrumental and Relational understanding (sometimes called procedural vs conceptual learning) [13]. Many learners (and pre-service teachers) recognize this distinction. At some point they were trapped within instrumental learning (just one fragile procedure and no options or alternatives). I find my students can describe when this instrumental learning produced lower and lower achievement from high school on into university – and finally failure by second year. While pre-service teachers in my classes often imagine that instrumental (procedural) learning must come first to then support later richer relational / conceptual learning, that is not what classroom research shows. In fact, it can take longer to move through instrumental learning then to relational, and students lose interest in the conceptual lessons. The result of a focus on instrumental learning can be: *less learning, and shallower learning*!

Recently, I was working with a few undergraduate math students trapped within instrumental learning. What started to work was engaging with multiple representations for every problem. Drawing on visual and spatial representations and associated reasoning was one of these added approaches that helped rebuild connections and conceptual understanding. One related general classroom strategy we talked about was "always check your solution" with a different approach, to build up the connections. A check is part of a complete solution in my courses, and a wrong answer without a check is two mistakes (where a failed check noted but without time to correct the solution is only one mistake). With this practice, the students are building up their internal conceptual network as well as a recognition that the answers 'make sense', to them, without relying on an outside 'expert' authority.

Young children are open, curious, and more capable than the curriculum recognizes. My goal is a mathematics classroom that engages students' questions, and curiosity. Mathematics classes should be one of the key places where students ask questions that matter to them. Of course, that means developing their insight into how these questions connect to mathematics, and how mathematics connects out to their (and our) world. That connectivity is real to those of us who work in applied (and applicable) mathematics and statistics. Making these connections and asking interesting questions is a learned skill, perhaps harder and more important than learning to answer the teacher's questions (which often evoke the response – "*when am I ever going to need this?*"). These connections can be found all around us – and need to be learned, noticed, and reconnected. Supporting this in the child requires teachers (and parents) to be curious, engaged, open to surprises, and confident enough to have students engage with mathematics and solving problems that are new to the teachers! Not only do we need "*mathematics for all*", but we need to spread the recognition that "*mathematics is everywhere*".

As we shift our classrooms in the direction of such connections, statistics becomes more central. Statistics is growing in importance, and in interest, to the students: at Harvard, 10 percent of a recent entering class chose statistics as a major in second year [14]! This shift is founded on a first year statistics class that engages curiosity and raises questions that matter. The New Zealand K-12 mathematics and statistics curriculum is another example: statistical inference, based on visual reasoning is included every year [15]. I predict that in the next two curriculum revisions, statistics will penetrate more deeply and more widely into the Ontario curriculum. That means in-service and pre-service teachers need professional development support. Statistics is not just applied probability – it is distinctive science (one of the reasons why many universities have a distinct Department of Statistics). Of course, it will not surprise people that statistical inference benefits from strong spatial/visual reasoning.

There is a lot of research out there that can impact our classrooms. The potential for change is enormous. I hope people follow up on some of these themes during the December 11-12 event hosted by the Ministry of Education.

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