INTRODUCTION: WHAT IS THIS RESOURCE ABOUT?

[Narrator:] This resource features primary and intermediate educators and their students focussing in on spatial reasoning and spatial visualization to support learning in the number sense and numeration strand. This focus on spatial reasoning emerged from the learning of two collaborative inquiry groups. A primary team of St. Agnes of Assisi in York Catholic District school Board and a junior intermediate team at Roselawn Public School, in York region District School Board. Both groups were exploring how to support students with learning disabilities in mathematics. As their facilitator, Connie Quadrini explains there were two important facets.

[Connie Quadrini:] The first facet was learning more about student profiles and the uniqueness of their profile, what strengths and what needs did those students have. But in addition we were also focussing on developing our content knowledge for teaching mathematics, which we know is critical.

[Narrator:] Their journeys prompted further investigation into spatial reasoning as educators discovered the power of using visual, concrete and digital representations to support not only students with learning disabilities but all students in math class. Primary students engage in spatial activities to represent quantities and to add and subtract, while intermediate students use spatial thinking to solve problems involving operations of fractions. While the focus is primary and intermediate the ideas and strategies related to spatial reasoning apply to all grades and can be easily adapted and used with students of all ages. Since spatial reasoning is not a separate area of study but rather permeates all of the math strands, it is important that we, as educators, can spatialize the curriculum by identifying where spatial reasoning can support the learning. There are examples of how the educators in this resource spatialize curriculum expectations and design their lesson plans. This is a good starting point when exploring this resource. Seven of the aspects identified in the Paying Attention to Spatial Reasoning document are explored in depth. These include using non-verbal reasoning, using proportional reasoning, composing and decomposing, manipulating objects, comparing objects, visualizing and scaling up or down. Although the specific aspects are featured individually the various clips reveal that there are strong interconnections among them, since spatial reasoning is a complex way of thinking and can draw on several aspects simultaneously. For each of the seven areas of focus there is an overall description as well as separate clips for the primary and intermediate grades which feature the spatial reasoning within the learning. There are also junior resources that relate to each of the spatial reasoning aspects, within these clips are strategies and activities that promote spatial reasoning. These align with the suggestions outlined in the Paying Attention to Spatial Reasoning document. Teachers are also featured engaging in spatial tasks since research shows that their comfort level with spatial reasoning is related to their students' growth in spatial skills. While junior students were not filmed for this resource, there are suggestions and resources for the junior grades that align with the primary and intermediate activities featured in this resource. Before viewing this resource it is recommended that educators solve the featured problems in order to gain greater insight into the mathematical thinking. These problems are available in the print resources.

INTRODUCTION: WHAT ARE SPATIAL REASONING AND VISUALIZATION?

[Narrator:] Recent research indicates that there is a strong link between spatial reasoning and mathematics learning and achievement. To take advantage of these findings and help improve student performance in math it is first necessary to understand what spatial reasoning is.

[Nora Newcombe:] Spatial reasoning is really I think two things. One is the kind of reasoning that we do when we try to navigate around the world and find our way. There's another kind of spatial reasoning that is kind of smaller scale, more focussed on objects, more thinking about what does this object look like if you turn it; that's often called mental rotation. Thinking about how long things are, how much they weigh, how much area there is. What they would look like if they were folded or bent or twisted or rotated. So those kinds of spatial reasoning I think are the ones that are at play when we think about math. So I think for each kind of mathematics and each kind of space there might be different relations. For each of these mathematical concepts we can think of the spatialization of it.

[Connie Quadrini:] The research suggests that spatial reasoning, in particular, spatial visualization is really important when it comes to supporting mathematics achievement and students feeling confident and proficient and having flexibility in their thinking.

[Narrator:] The Ministry's "Paying Attention to Spatial Reasoning" document describes spatial visualization as, "Using our imagination to generate, retain, retrieve and transform well-structured visual images, sometimes referred to as thinking with the mind's eye."

[Sue Ball:] For me as a psychologist when I think of what's required for spatial visualization there are many different skills that are required- non-verbal reasoning, part-to-whole concepts, the ability to transform to see objects and to be able to transform and manipulate. Those are all essential skills, but that's not just one skill.

[Connie Quadrini:] Part of that work is having the teachers themselves practice what it means to visualize, and it's encouraging them and inspiring them to bring that--those opportunities to their students.

One thing that we know from research is that spatial visualization, spatial reasoning, is malleable. How exciting is that for us?

Helping them create mental images can come through the use of concrete materials, digital kinds of resources, providing the gateway for them to be able to then visualize because they have those experiences. It's helping it go into their long-term memory. They're immediately drawing on those prior experiences, because they've been memorable.

SUPPORTING STUDENTS WITH SPECIAL NEEDS

[Narrator:] As the "Paying Attention to Spatial Reasoning" document highlights there is still much research to be done around students with special needs in math. We do know however, that strength in spatial ability is related to success in math and that weakness in spatial domains can negatively affect math performance.

[Connie Quadrini:] The work that we've done around better understanding the profile of students with learning disabilities - we are always looking at their strengths and their needs. What we're also trying to do through this collaborative inquiry is have educators think about leveraging those students' strengths and responding to the needs. So very often when students are struggling say when it comes to verbal comprehension, whether it's oral forms of communication or receptive or being able to read, being able to make sense of kind of written text, we're finding that in mathematics through the use of the tools many of these students actually have strengths when it comes to visual representations, that perceptual reasoning. And many of them have strengths also in visualization and spatial visualization.

[Narrator:] While spatial visualization may be a strength for some students with special needs it can be an area of need for others. A significant finding is that the tools have been beneficial for supporting students with strengths or needs in spatial visualization.

[Dr. Sue Ball:] The visualization strategy; for a student with a learning disability that may be difficult. That may be something that they have great difficulty with. That may be their area of need. So the opportunity to have math tools to work with; it's good for all, but for students with a learning disability, it's necessary for them to have access to those tools to be able to demonstrate the learning that they're capable of.

[Teacher:] Something like this with the manipulatives, the way you set up the lesson, who you're trying to reach and how you're tapping in on all their different strengths, for me that really proved to me like differentiated instruction. Everybody is getting some part of it, and everybody's communicating back whichever way they can. So you're still understanding - oh, they get this part; they're still working on this. They're past that point; they need to be challenged past that point. So for me it really opened up that whole differentiating instruction kind of learning.

[Connie Quadrini:] One thing that we know from research is that spatial visualization, spatial reasoning is malleable; how exciting is that for us? I remember asking a researcher and saying you know what does that mean for students with learning disabilities, and the researcher said, 'well we have no reason to believe otherwise that this malleability, this potential for it to develop is there not only for students in general, but in--but also for students who may have a learning disability.'

[Dr. Sue Ball:] As a psychologist I'm so excited about this work and how we can work to ensure that our reports and our assessments are fitting in with math knowledge and understanding that the recommendations are valuable, that they--that they're not just accommodations for learners with a learning disability, but supportive of instructional strategies that can really help show that student's potential. If that understanding is there of how they learn with strategies that support using their strengths, then they'll be much--as we've seen with this collaborative they'll be much more successful, which will have an impact as we've seen with this project on how students are feeling on the success that they're able to achieve and how it impacts on their well-being, and it impacts on their motivation, and their engagement.

COMPOSING AND DECOMPOSING - OVERALL DESCRIPTION

[Narrator:] The Paying Attention to Spatial Reasoning document identifies composing and decomposing as important aspects of spatial reasoning and gives geometric examples of combining and taking apart shapes in order to make new shapes. Composing and decomposing are also important in number sense and numeration. As students use concrete materials to represent quantity of numbers. Students often compose and decompose in terms of friendly numbers like 10. Junior and intermediate students decompose and compose with various concrete materials and tools in order to understand quantity of numbers such as fractions. Composing and decomposing with concrete materials and tools also helps students conceptually understand operations of numbers. For example, when learning multiplication strategies for 4 times 7, students can create and then decompose the representations in more familiar ways such as 4 groups of 5 and 4 groups of 2. They can then recompose their products in order to find the total of 28. The tools create the mental images that students can later retrieve when solving other math problems. Similarly, as grade seven students add and subtract fractions by composing and decomposing with a variety of tools, they can visually see what is happening when various operations are applied and form powerful mental images. Through this learning process, it is also important to link these visual representations to their numerical form, so students can make the connections. In this resource, students in primary grades compose and decompose with whole number rods in order to add and subtract, while intermediate students compose and decompose with relational rods and digital tools to perform operations with fractions. Ideas are also given for composition and decomposition of numbers in the junior grade.

COMPOSING AND DECOMPOSING - PRIMARY - UNDERSTANDING QUANTITY

[Connie Quadrini:] I began working with the St. Agnes of Assisi Group, the primary team. As we began to work together and I had them solve several different kinds of problems I invited them to use the rods or whole number rods in the context of primary. And what was fascinating is we got into ideas such as composing and decomposing of number through the use of the rods, and how the rods actually provided opportunities for students to be able to think very spatially about numbers.

[Teacher:] I think my ahh-ha moment came using the rods just being able to compose and decompose them. Because we were always teaching the kids about the number ten, and ten is important, but not realizing that within the ten there is all different combinations these kids could be thinking of and making up, and using the rods to do that.

[Elisa Aquino:] And maybe not recognizing that the kids themselves didn't know how to compose and decompose those numbers, and so they're trying to teach them a strategy when they didn't even have a sense of what the unit was. I think some kids were at a disadvantage not being able to visualize the numbers in a different way, and this allows them to see every number as a quantity and not just as an abstract number.

[Student:] So that's 10 plus, and then you can add 1 more then that's 21. And then you can add eight so that's--this is ten. That's where I had the need for this one, and then this is 21, 22.

[Student:] Are we supposed to make 28, right?

[Student:] Twenty-three, 24.

[Student:] So this and this--sorry--^M00:01:59 here is 10, and then this is 20, because this is a 9, Robert. If you add the 1 rod this is ^M00:02:11 20, and then if you add these 8, the 1's this is 10, 20, 21, 22, 23, 24, 25, 26, 27, and 28.

[Rosanna Cristello:] It just helps to know that what actually a number means.

[Connie Quadrini:] As an educator the work that you're doing and how you're supporting kids in that scene recognizing the strategies of decomposing and recomposing is helping them think even with larger quantities, and that's the goal of creating the visual or that visualization and then paring it up numerically is helping kids connect.

[Student:] So that' s10 plus 10, plus 9 plus 1 that's 10, and 10 plus 10 plus 10, plus 5 plus 1 that's 6. So 5 plus 5 plus 10 plus 9 plus 1 plus 5 plus 1 equals 36.

[Connie Quadrini] So what--the meaning that the numeric now has when there's a visual representation in the mind and they've reasoned through it.

COMPOSING AND DECOMPOSING - PRIMARY - UNDERSTANDING OPERATIONS

[Connie Quadrini:] The primary group, they were curious about, you know, could the rods be used as a way to support operations, because they realized their students were struggling with operations, the idea of regrouping, you know, break a 10 apart, in order to have ones and subtract. So we began thinking about the relational rods and how they might be used to explore operation.

[Ida-Lee Tessa:] After creating 36 they wanted to know what the extension was to get to 57.

[Student:] So this remains 57.

[Elisa Aquino:] Yes and then?

[Student:] Then we have add it all up.

[Elisa Aquino:] Yes.

[Student:] And the answer we got was 57.

[Elisa Aquino:] That's the total for the first one, right?

[Student:] Yes.

[Elisa Aquino:] And then you did what? You made what?

[Student:] Um 36.

[Elisa Aquino:] Yes. And--

[Student:] He compared it like as an extension.

[Elisa Aquino:] As an extension. That's right. They wanted to find the extension. That's what they were looking for. Now they struggled to find that, and they needed some support getting there, but they recognized that there was something that they had to find the emptiness. They had to find that extension and they knew that was where the answer was.

[Ida-Lee Tessa:] Yeah.

[Elisa Aquino:] Because then he was going to fill in the pieces using leftover rods. Yeah. Very nice. He could spatially see that.

[Connie Quadrini:] So what you're describing is how the rods help them spatially think about that missing quantity.

[Elisa Aquino:] Yes. Because that's when he used that word extension.

[Cristina Manzone:] I was just so impressed with all of the different strategies that they came up with. Like it blew my mind, like that group of three--

[Student:] We subtracted well seven and six that equals the one. Then we subtracted 30 from 50 and we got 21. How we got the ways of 21.

[Christina Manzone:] What they did is they subtracted the ones and they got one left and then they subtracted the ten's number using the rods. And they had 20 left and I was like "Yes. I see." I didn't even see that.

[Connie Quadrini:] So we can certainly document and annotate through the visuals in a recording, but we can also pair that up with a documentation that's numeric.

[Student:] When you take away--

[Rosanna Cristello:] So what Claire is saying is they put their pencil here, to show that these parts were the same, but this is the part that was the difference. That's where the nine minus six came from. And that's the three. If you count the white units, one, two, three. That's where it was different. And then Claire said she-- they added on. So Claire, what were the numbers that you added on to figure out how much greater 57 is?

[Student:] We added three, plus one, plus nine, plus one, and plus seven.

[Rosanna Cristello:] And what did that equal all together? What was the total?

[Student:] Twenty one.

[Rosanna Cristello:] Twenty one.

[Connie Quadrini:] And it's a way of being able to document or annotate relationships, right? That being able to compose and decompose and what that means in the context of the problem we gave them today.

COMPOSING AND DECOMPOSING - INTERMEDIATE - UNDERSTANDING QUANTITY

[Connie Quadrini:] How might we even solve it using the rods?

[Linda Macris:] Okay, well, I think the first thing is that you have to figure out what hole you can use and part of that is going to be students figuring out the relationship between the rods and then they could try adding in smaller pieces to find out what exactly does this represent. And go, okay, well, so those match up but what does that mean? So I have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 of the smaller pieces so this would be, each one would be 1/10. So I have 10/10 in total and if I'm looking at this problem and I know that I need 1/5 or 2/5 then I would start saying what can I match up, any other pieces, and make relationships there. So if I take this piece I know that now that that is the same as two 1/10.

[Connie Quadrini:] Mmhm.

[Linda Macris:] And if I take more of them, and because I know that it matches up here I know that I'll be able to create a relationship between the red and the orange as well.

[Connie Quadrini:] Nice, yep.

[Linda Macris:] Okay? So now I can take, so now I've got 1, 2, 3, 4, 5 of those so each of these would be 1/5 and I know that I need 1/5 or 2/5 here. I'm going to take away these ones. That was what I used to help figure out the relationships. I have 1 whole and I know that I need to use 2/5 of my whole to create a decoration and there's my 2/5 for 1 decoration and 2/5 for another decoration and I have 1/5 here.

[Connie Quadrini:] You actually use the tools and you took this red one, which was 2/10, and then you could have iterated that so that you had 5 of them, so it was like you know, five 1/5 and that's your unit fraction, which is really cool. The other thing that you did is you started to actually unitize two 1/5 together so this idea of taking a fifth and a fifth, that composing to make 2/5.

[Student:] For 2/5 I'm going to first start off by showing it as 2/5 before I show it as anything else.

[Connie Quadrini:] With the fraction strips they were able to kind of, they were able to unitize two 1/5 using that equivalency bar so it's neat that they were able to see that they needed two 1/5 units and that, again, when they drag that out they could see that in relation to the whole.

[Student:] Since there wasn't any 2/5 I used like two 1/5 to make it.

[Linda Macris:] Why are you using the two 1/5s?

[Student:] Well because that's how much a ribbon decoration uses.

[Linda Macris:] Okay.

[Student:] Two of these 2/5 pieces would create 1 piece of decoration. Since there were 2 and 2 and a half pieces every metre but then there were like half pieces too, so then 2 of the halves that would make another decoration but then there would be another half extra. So in the end like there will be 7 1/2 decorations that would be made.

COMPOSING AND DECOMPOSING - INTERMEDIATE - UNDERSTANDING OPERATIONS

[Linda Macris:] So I have my three metres of ribbon and now I've got groups of 2/5 to create each decoration. I have one decoration, two decorations, three decorations, four decorations, five decorations, six decorations, and each of those decorations is the 2/5. And then I have these ones which were part of the original metres and I have three of them, so three 1/5. If I put two of those together, then I can create another decoration.

[Connie Quadrini:] This was really interesting here when you went from the left over 1/5. So you actually took a 1/5 from your second whole and combined it with a 1/5 of your third whole, so again that notion like what's so evident within that solution that you created is this idea of identifying the unit fraction with the tool but then composing the fractions, the unit fractions, to create a new unit of 2/5, but then decomposing them, recomposing them again, based on left overs. We could really do a nice job in the lesson of annotating that. Like this idea of a number sentence. We were talking about that earlier of taking the number of 2/5 but then those 1/5 and then how those two come together to create another unit of 2/5.

[Student:] Okay. If you count like two of the 1/5 of the 2/5 – 2/5 with the – yeah 2/5 with one metre. Then if you keep going, if you keep counting up like that.

[Linda Macris:] So what Andia is circling is what she did, she repeatedly iterated and we've used that word in math class, that means repeat, we've iterated the 2/5 again, and again, and again.

[Connie Quadrini:] We kind of see up here this taking like unitizing two 1/5 and then taking that and composing it to create 14/5.

[Linda Macris:] All right, can we go back to the one on the top, where you said you had 2/5 plus 2/5 plus 1/5, and what happens there when you get to the 2/5 plus 2/5 plus 1/5.

[Student:] It's one whole.

[Linda Macris:] One whole or 5/5.

[Student:] That's one metre.

[Linda Macris:] So what you're doing is you're putting them together, you're composing, you're adding them together for one whole.

[Connie Quadrini:] I quite liked how at one point when the second pair were presenting and they were talking about the 2/5, 2/5, and then there was that extra fifth, and how they had talked about the 5/5, it just set us up beautifully now to look at a different kind of composing.

[Linda Macris:] Right, so instead of composing those two, they took those and composed it in to the one wholes.

[Connie Quadrini:] Yes. And then a different kind of composing and to actually create the

14/5.

[Linda Macris:] And then we did have the one student who took the total number of fifths there were in the three metres, and then subtracted out the 2/5 again and again and again. The whole idea of decomposing and composing. The other thing is just the relationships that can be made at that point between in this case, I'm thinking the operations and fractions as an inverse operations where you no longer think of addition and subtraction, multiplication and division as separate entities.

MANIPULATING OBJECTS - OVERALL DESCRIPTION

[Dr. Sue Ball:] The opportunity to have the math tools, so to have the fractions strips. To have the relational rods. To have the concrete manipulatives can support a lot of learning for a significant number of students.

[Connie Quadrini:] What kinds of interactions with concrete tools, digital tools will support students in developing that mental picture, which will support them then with visualization.

[Speaker:] The Paying Attention to Spatial Reasoning document emphasizes that manipulatives are not just used to communicate or show representations. But are the tools with which the problems are solved. Manipulatives help students move between concrete representations and abstract ideas by allowing them to visually understand and internalize abstract concepts. As students see their concrete representations and test their ideas through tactile experiences, they can also uncover their own misconceptions. Educators also need to solve problems with manipulatives in order to understand how the materials can reveal important mathematics.

[Janine Franklin:] Well, this year we were looking at fractions within the realm of spatial reasoning. And I know for myself sometimes we go to our default, which might be an area model of fractions. But by challenging ourselves to think about other models, it's actually drawn new connections to the mathematics content. This learning opportunity really gave us the opportunity to understand the student in front of us, try to unpack the thinking and the visuals that they have in their head. But then start to understand, well, what prompts might we actually need to be asking or questions to help them explore what else that could look like. So that, as they're developing their mathematics, it will push them to new places.

[Sandra Fraser:] For me it has been important that I should be able to figure it out using multiple tools. Multiple, you know, visualizing it in different ways, representing it visually. The biggest change for me is when I go in and see the students working with the tools. I'm able to sort of support more than I would have been able to in the past.

[Connie Quadrini:] We've been able to see certain students who through the use of the tool are able to make significant connections. And visualization may not necessarily be their strength to leverage. So it's telling us that the work we're doing with the tools and the work we're doing around bringing attention to spatial visualization within our work is making a difference for them as well. There's still lots of learning to do in that regard, but we know that we're onto something.

[Speaker:] In this resource, as educators use manipulatives to anticipate student thinking, they learn more about the tool's features and how they support conceptual understandings of the math. Primary and intermediate students use the tools to solve number sense problems. And then share how they have been helpful in their learning. In two clips, students uncover their own misconceptions by using the manipulatives. Ideas are also given for using concrete and digital tools to support development of number sense in the junior grades.

MANIPULATING OBJECTS - PRIMARY - HOW TOOLS SUPPORT

[Teacher:] I like the rods, too, and I like the proving part because a lot of kids are not, their strength is not evaluating their own work. But with the rods, the fact that they can actually use the rods, they can self-evaluate. And, and I like that.

[Student 1:] This is 8. And another one is 8, and 16 - and yellow is 5?

[Student 2:] This is 4.

[Student 2:] Wait, that's 3. ^m00:00:33 This is 4.

[Student 1:] Okay, done. ^M00:00:36 I did my idea. Now do yours.

[Student 2:] Mine is just, like-

[Student 1:] 10, 10, 8.

[Student 2:] 2 of these, and, which is 8.

[Student 1:] 10, wait. Are these 8s? ^E00:00:50 ^B00:00:53 Wait, 8, 16.

[Student 2:] Yeah.

[Student 1:] Wait, this doesn't make sense. How is yours longer than mine?

[Student 2:] I don't know.

[Student 1:] They should be both equal size.

[Student 2:] I think I know why. Because these go like this. You used smaller ones than me. Like this equals. This equals 20. If I add this to this.

[Student 1:] Mine still equals 28. This is.

[Student 2:] Yeah.

[Student 1:] 8, 16. No, 8, 16. I need another 8. ^e00:01:31 ^B00:01:32 8, 16, 24 and 4.

[Student 2:] Yeah, there.

[Student 1:] There's our mistake.

MANIPULATING OBJECTS - PRIMARY - TEACHERS USING TOOLS

[Connie Quadrini:] We in the primary classroom made the decision to have students work primarily with the rods. Or whole number rods in the context of primary and we use the Mathie tools, the whole number rods tool from Mathies, as a way for students to be able to share their thinking and allow for other students to see the concrete representations. Okay. So we're going to take some rods out and we're going to try this out together.

[Speaker:] In order to understand and anticipate how students solve problems using the tools teachers must first develop their own comfort level with various manipulatives. This includes being aware of the available concrete and digital tools and becoming proficient with using them while solving problems. Through hands on experiences educators can not only investigate how fundamental concepts are revealed through the structure of the tools but they can also further develop their own spatial reasoning skills.

[Ida-Lee Tessa:] First I would make the 57. Yeah. So since there's no more orange then I would do it this way.

[Nancy Ventrella:] I'm going to make 36.

[Ida Lee Tessa:] Fifty right?

[Nancy Ventrella:] I'm going to see how many more I need to make 57.

[Ida-Lee Tessa:] Okay. And then -- the yellow yeah.

[Nancy Ventrella:] Or else do you know where the 7 is?

[Ida-Lee Tess:] This is 5 and that's the 2. Okay good. And then I would need the 36. Right so you need this. This is the difference right here.

[Nancy Ventrella:] But can I do that?

[Ida-Lee Tessa:] Right? What's the difference? 21.

[Nancy Ventrella:] Yeah.

[Ida-Lee Tessa:] But I don't know that they're going to see it this way. I don't know that they'll get a purple. They'll figure out the four like this. So how many are left over right? They might I think. They'll do that and say okay this is a four. So this is a four.

[Nancy Ventrella:] So they might even do it this way with all the ones.

[Ida-Lee Tessa:] They might.

[Nancy Ventrella:] And count. But then after they might realize doing it like this, right? Seven. They take that, yeah.

[Ida-Lee Tessa:] And the 3. They take the 3 out -

[Ida-Lee Tessa:] And they get another rod.

[Nancy Ventrella:] And then another orange rod so maybe they can get this rod and then they'll see this is what is left over.

[Christina Manzone:] It's easier for me to add than it is to subtract. So I would actually add to solve this problem. So we are -- the starting number is 36 so I would make 36 so 10, 20, 30. Was it 36? Yeah. 35, 36 and then I want to know how many more I would need to make 57. So I was at 36, 46, 56, 57. So I would just -- this number would give me my answer. So actually if you slid that over, those ones, right? You can see how the combining happened and then you could even reorganize right. You could take your five and one and reorganize. So very nice. And you know, my number sentence for this would be an addition. The number sentence. So 36 plus the 21 would give me the 57 with 21 being that magic number.

MANIPULATING OBJECTS - PRIMARY - STUDENTS USING TOOLS

[Connie Quadrini:] One of the things that the teachers realized after using the rods is that it was right there. So many of them said, “I can see it.” The rods actually helped them create mental images in their mind and they were really excited about trying that with students.

[Connie Quadrini:] Show me where your 57 is.

[Student:] I have 57. Then we minus the 3 because it's 36 and then took away the 7 ^M00:00:41 and because it's a 6, it left 1 out so we just put a 1 here.

[Connie Quadrini:] Fantastic, so you took away, you subtracted. Good idea.

[Connie Quadrini:] Those experiences about the relationships when it comes to number create those mental images. So again that spatial visualization through the use of the rods is helping them be able to create those mental images. It is helping it go into their long term memory and then that's helping them to retrieve it later.

[Student:] Whenever I don't really know which one is the number I just count. Like I start at the 5 either go down or up because I know 5 is the middle.

[Connie Quadrini:] They are immediately drawing on those prior experiences because they have been memorable.

[Student:] They actually help me to like count on so like I put that there and then I kept putting rods to equal that and then I counted all of them and they equal the number.

[Connie Quadrini:] A big part of that is how numeric then matches up with concrete tools.

[Student:] Add 2 more rods and a 1 rod and makes 57. And so it has -- we used 5, 10 rods and we used 1, 6 rods and 1, 1 rod.

[Connie Quadrini:] What was different about those tools, those Mathie tools? Then these rods here?

[Student:] Well the rods on the screen had the numbers on them.

[Connie Quadrini:] And how was that helpful?

[Student:] Because we know what numbers it is. Like the 10 rod, the 9 rod, the 8 rod.

[Connie Quadrini:] What the students really enjoyed, the primary students really enjoyed was the fact that we could turn on the numbers and turn them off. So students could actually match up the numeric representation for the rod, the value or the quantity of the rod with the actual rod. And that was a great support for them to think about their number sentences and how the number sentences would match their rods. What would you say has been kind of either eye opening for you or has been an aha moment for you around spatial visualization through the use of these tools, the Mathie tools?

[Speaker:] I think they help all or every level. But for me I noticed that they all benefit from using the manipulatives.

MANIPULATING OBJECTS - INTERMEDIATE - HOW TOOLS CAN SUPPORT STUDENTS TRADITIONAL UNDERSTANDING

[Teacher:] I like the rods, too, and I like the proving part because a lot of kids are not, their strength is not evaluating their own work. But with the rods, the fact that they can actually use the rods, they can self-evaluate. And, and I like that.

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[Student 2:] I think I know why. Because these go like this. You used smaller ones than me. Like this equals. This equals 20. If I add this to this.

[Student 1:] Mine still equals 28. This is.

[Student 2:] Yeah.

[Student 1:] 8, 16. No, 8, 16. I need another 8. ^e00:01:31 ^B00:01:32 8, 16, 24 and 4.

[Student 2:] Yeah, there.

[Student 1:] There's our mistake.

MANIPULATING OBJECTS - INTERMEDIATE - STUDENTS USING TOOLS

[Connie Quadrini:] The use of the tools is certainly a strategy, like they're creating a model. So, we might want to emphasize what particular model they're, or what particular tool they're using. So, like, this happens to be fraction strips but we may have them using relational rods.

[Student 1:] I think if I have a visual representation, it's easier than just writing my math problem. Because the visual is a really concrete description of your answer. For me, I'm just very manipulative based because if I have concrete bars or, like, sticks that I can show numbers with, it's easier for me to make a, like a number line with concrete tools instead of numbers because I can manipulate it with my hands and it's more, it's more interactive.

[Student 2:] So, I like when, learning with manipulatives to actually see the problem. And I also like when we share ideas because sometimes I have a strategy that doesn't really work for me and I can see other people's strategies and sometimes that strategy might work better than me.

[Student 3:] These also help you a lot too because it's kind of like this. Like, you just, when you, like, display everything, it just helps you visualize.

[Connie Quadrini:] It's given me a real opportunity to think about spatial visualization and spatial reasoning through the lens of those digital tools. For the teachers who have been using the Mathies digital tools, it's their recognizing the power of the features of the tools. For example, the fraction strips. It gives the opportunities for students to be able to look at the unit fraction and the power of the unit fraction, being able to iterate the unit fraction to, you know, create the whole or, or create a fraction greater than a whole.

[Student 4:] I'd say that they help because if you can use the tool and you can use the drag tool to see the equivalent fractions and you can drag each individual ones, then you can kind of see more clearly than if you're writing with pen and paper. Because that's really rough whereas this is very distinct and you can see it better.

[Student 5:] I love going on the computer, going to, like, the Mathies website, using the fraction strip tool that we used today. I like the magnet mode. It allows you to straighten up everything, like, making it like this instead of having to go with a ruler and straighten it up like this and then push them all toward the ruler. All you have to do is drag it and it sticks to it.

[Student 6:] You can see how they all relate to each other for a whole and that really shows what you're working with.

[Connie Quadrini:] With some of the learning we've been doing around, you know, how the tools can actually support us as educators and students and thinking about what does that image look like, and then how does that pair up numerically, for example, naming the fractions.

MANIPULATING OBJECTS - INTERMEDIATE - TEACHERS USING TOOLS

[Narrator:] In order to anticipate and understand how students use tools to solve problems, teachers must first develop their own comfort level with various manipulatives. This includes being aware of the available concrete and digital tools and becoming proficient with using them when solving problems. Through hands on experiences, educators can not only investigate how fundamental concepts are revealed through the structure of the tools but they can also further develop their own spatial reasoning skills.

[Connie Quadrini:] The role of spatial visualization, how that plays out within concrete and visual and digital representations and what that means for their numeric forms, how do I bring attention to these important ideas, important connections, to the educators? So the problem we were thinking about was the ribbon task, right? You have 3 metres of ribbon. Each decoration needs 2/5 of a metre. How many decorations can you make? So that might be helpful for us to also think through what might be the different ways that kids will solve it, even how we will solve it.

[Linda Macris:] I have one whole and I know that I need to use 2/5 of my whole to create a decoration and there's my 2/5 for one decoration and 2/5 for another decoration and I have 1/5 here. But I know that I have 3 metres of a ribbon. So if I continue with this pattern I know that I can have my next metre there. And I'm going to show that I can go there's another 2/5 and another 2/5 and I have one left over. And I have one more metre of ribbon. There's my metre, my third metre of ribbon. And I'll separate it out again. If I put two of those together then I can create another decoration. So in total I have 1, 2, 3, 4, 5, 6, 7 decorations with 1/5 of a metre left over.

[Narrator:] The Ontario Ministry of Education has developed a number of digital resources called Mathies Learning Tools that both educators and students can use when teaching and learning mathematics. Educational programmers have strategically embedded structures that make mathematical concepts evident and features that support the mathematical thinking of all students with diverse strengths and needs. The digital tools are available at no cost, are free of advertising and can be accessed at the site on the screen.

[Connie Quadrini:] I took the actual Swift File from the Mathies Website. I saved it. And then I just incorporated it into an interactive white board file so we can kind of interact with it but then we can also use the text feature if we want to annotate anything. And in addition we can record it as well, which is kind of neat. Let's try the next solution. So what are you thinking?

[Speaker:] So we start off with groups of 5 1/5 and we bring over 3 groups of 5 1/5.Three groups of 5 1/5.

[Speaker:] So I've got one group of 5/5.

[Speaker:] So this time we're starting off with 1/5 instead of starting off with wholes.

[Speaker:] Ok two sets of 5/5 and three, ok? So now?

[Speaker:] And then as we use groups of two 1/5 for the decorations we subtract them from the three wholes.

[Speaker:] Ok so that means I need to take two 1/5 at a time. So that's --

[Speaker:] Repeated subtraction.

[Speaker:] Ok so it'd be something like 15/5. Take away.

[Speaker:] Two fifths.

[Speaker:] Ok so let's take away another 2/5. Oh kind of group them together. So there's three 2/5.

[Speaker:] Four 2/5.

[Speaker:] There's four 2/5. There's five 2/5. There's six 2/5. Oh I like to group that together.

[Speaker:] Or can we use the 1/5 and 1/5?

[Speaker:] Yeah nice idea.

[Speaker:] And then to create wholes.

[Connie Quadrini:] So there's another set of 2/5. Oh I guess you're right. I'm going to have to break that up. So one here and then one here. So this is pretty fascinating actually. So that repeated subtraction actually builds up to the three wholes again that you could see with the tool. As we deepen our own content knowledge, that's helped me to appreciate much more the connections between concrete and visual and digital representations of fractions and how they pair up numerically.

NON-VERBAL REASONING - OVERALL DESCRIPTION

[Narrator:] Nonverbal reasoning is also part of spatial reasoning. It involves making

sense of information and solving problems without the use of language. It plays a significant role in learning mathematics so students form concepts in their minds by using visuals, like pictures and diagrams, and by engaging in hands-on experiences with concrete materials. In number sense and numeration, visual representations can support students in making sense of quantity and operations and can help them remember these concepts through the mental images they have created.
^M00:00:39:15
Since nonverbal reasoning is not as relied upon language, it's important that educators pay attention to students' gestures and their actions with concrete materials in order to gain greater insight into what students are thinking and what is understood. It is also important to link mathematical language and vocabulary to nonverbal reasoning so students can communicate their ideas to others. In this resource, recent research about the power of gestures and teaching and learning math is discussed. Primary and intermediate teachers observe and analyse their students' gestures and actions with the tools to glean more information about their mathematical understanding. In the consolidation, teachers provide language with the nonverbal reasoning so math concepts can be shared and discussed in class. Related resources for the junior grades are included.

NON-VERBAL REASONING - LINKING LANGUAGE TO NON-VERBAL REASONING

[Connie Quadrini:] We do want students to become proficient and fluent with operations. But the way that they become fluent is if they've had those memorable experiences and, in our case, the memorable experiences were anchored into the use of the tools and the visual and mental imagines that were being created as a result of that.

[Ida-Lee Tessa:] Well, I also think that it lends to the kids that also have trouble communicating in written form. You know, when they're able to dialogue and explain it to somebody.

[Connie Quadrini:] Then how much did you take away?

[Student:] Two rods and one green rod.

[Connie Quadrini:] How much is that?

[Student:] Twenty one.

[Connie Quadrini:] Okay. So, now I want you to try it again, and I want you to each take a turn explaining it because it's really important you know exactly what you did.

[Student:] This is subtraction.

[Connie Quadrini:] But you've got to make sure that you're very clear, so that other students understand. So you take a turn and then you take a turn, okay?

[Ida-Lee Tessa:] It helps them to show it and explain it using the malipulatives, so you actually get to capture, do they understand the concept?

[Connie Quadrini:] As kids are describing the way they've done it, you can help them kind of -- it's the talk with visualizing so we started with 36, and we're adding on so that's becoming longer, you know. So they're getting that idea of the space, that length, that quantity of coming up to 57, right? That will really support the kids as well because you're helping them, again, with that mental imagine and how that connects numerically. The consolidation is having students be able to talk about what was important and bring those ideas out so that we know that everybody has a sense of what was really important about the lesson today. And, in fact, those ideas that emerged came through the use of the tools and the rods. Students not only brought out the strategies, but then had visual anchors to them.

[Rosanna Cristello:] Can you explain to everyone what you did?

[Student:] First of all, we started off with the number 57 made by rods.

[Rosanna Cristello:] Okay. And what rods did you use to make the number 57?

[Student:] Five -- you can say this one.

[Student:] Five orange rods, and one black rod.

[Rosanna Cristello:] And one black rod? Okay.

[Student:] We subtracted three orange rods.

[Rosanna Cristello:] So you subtracted three orange rods, and what number did that equal?

[Student:] That equaled 27. But then we took away the 7 and put a, and added a 1 because 7 minus 6 equals 1.

[Student:] And then that goes with 21.

[Student:] 57 minus 36 equals 21.

[Connie Quadrini:] So the rods themselves are allowing students to come with very, up with very diverse strategies and compensation is one of them. Knowing that if I remove this, I need to bring one back to be able to make up the quality that I need for that seven. The co-teachers introduced the terms adding on strategy or comparing strategy or a subtraction strategy. They could see when a student was using a compensation strategy, and they could name it. We can introduce terminology, but for some of our students who actually need those visual representations, it actually helps them to better understand the concepts. That helps give them language.

[Connie Quadrini:] For what they're doing.

[Cristina Manzone:] And having something to refer back to for future reference, right?

[Connie Quadrini:] Being able to link the gestures and the actions with the tools and the conversation and the naming of the math thinking, that approach, is really going to support kids in many different ways so those who, you know, have strengths in verbal, those who are more non-verbal, those who have strengths in, you know, spatial reasoning. All of that is going to give lots of entry points for children.

NON-VERBAL REASONING - PRIMARY - ACTIONS WITH THE TOOLS

>> We're finding that in mathematics, through the use of the tools, although they may not use as much verbal language, many of these students actually have strengths when it comes to visual representations, perceptual reasoning. And many of them have strengths also in visualization and spatial visualization. We're starting to pay attention more to the actual actions with the tools. Were they are actually doing this with their hands? So there's the gestures right. They're realizing so they could be saying a ten and another ten and a one. So it's important. It's great that you notice that because that gesture is telling you they're understanding there's that extra that they need. That adding it up, that adding on to be able to fill in the extra space. And again it's a lovely link to the paying attention to the actions with the tools. A taking away is much different than an adding on. First as a comparison and then matching up. Linking together the gestures, which is non-verbal, the actions with the tools are really helping kids have different entry points in to those memorable experiences. One of the things that will be important for the co-teachers then is to think, have children, when they start doing their number sentence after, we need to ask them how does your number sentence match what you did. It's just not the final representation. Because actions can look very different when you're comparing. You could be removing. You could be adding. You could be rearranging. You could be decomposing. ^E00:01:55:29 ^B00:01:58 There could be a lot of different actions. And the actions actually, when you create your number sentence, are very different. As you check in with kids, show me what you did, how does that match what you did. Oh, I actually see a - you actually took some rods away. You know, like so helping them to make those connections from the action that they did to their number sentence.

>> Yeah. 10, 20, 30, 40, 50. And then -

>> Add 7. 7. So what's 7.

>> 1, 2, 3, 4, 5, 6, 7.

>> OK that, and a black.

>> Yeah. That's 57.

>> So are we going to like write on the sticky notes? No, but not yet. Okay, so this is here. So it's 21, because, yeah. This is eight.

>> That's 6.

>> That's 6?

>> This is 9.

>> 7, 8, 9.

>> That's 3.

>> Yeah.

>> 10 and 4 and then 14. 14 plus 7 is 21. so 21. Let's think about the sentence again.

>> So you add 10, 20, 30

>> And then we put 57 subtract 36 equals 21 and that was our end.

>> Okay, can you, they did something really important I think. Can you tell Ms. Quadrini what you did with your pencil? Explain that.

>> We put our pencil to divide where it ended so that we knew that six minus six, yeah, nine minus six equals three. Plus the other white one and then we added on from there.

>> What were the numbers that you added to figure out how much greater 57 is?

>> We added three plus one plus nine plus one plus seven.

>> And what did that equal all together, what was the total?

>> 21.

>> 21.

>> You know what was interesting? We thought that they were going to have a really hard time at writing a matching number sentence to go with it. Once they saw the visual they were able to very quickly come up with a number sentence that matched. Whereas if we were to do the reverse, just asked them to write a number sentence to solve a problem, I think that they would've really, really struggled with this.

>> So before I added these two together it was first 21 and 36. Then when I added it together I figured out that it was the same length. ^M00:04:51 And normally when it's the same length, same number. 57. My number sentence is 36 plus 21 equals 57.

NON-VERBAL REASONING - INTERMEDIATE - SUPPORTING NON-VERBAL REASONING

[Dr. Sue Ball:] For certain students, they will require that concrete manipulative. The ability to take that fraction and be able to place it and see the relationship between the fractions and how it adds and when it's greater than a whole. That might be necessary for that student to be able to show and demonstrate the knowledge that they have. For another student it might be work on the, with Mathies, a computerized tool that can support all of our learning to be able to show the potential that they have.

[Connie Quadrini:] I also wonder that, if they, if they're practising what that looks like or sounds like, when they go to share with a partner, they could certainly verbalize. But maybe the tools might be there to be able to show it. And the tools might actually help them in being able to communicate that picture that's in their mind.

[Linda Macris:] I think giving them the variety, the options.

[Connie Quadrini:] Yeah.

[Linda Macris:] Is a good idea.

[Connie Quadrini:] Yeah.

[Linda Macris:] Because some of them will have this beautifully laid out in their mind, but they won't be able to articulate it very well.

[Connie Quadrini:] Yeah.

[Student:] I thought of it as, looking at, took another 2/5. ^M00:01:17 So you basically had it like that. See how I have 2/5 and 2/5. And then you have the last 1/5. ^M00:01:26 So each one of those would a decoration, and that's obviously half.

[Connie Quadrini:] In the minds on section, as we were kind of walking around, was there anything that you heard that you found interesting based on that activity of having them visualize the task?

[Linda Macris:] What they had in their head, they immediately were able to transfer into a form that other people can see.

[Connie Quadrini:] Quite a few of them actually grabbed the tools to be able to represent that mental image. So that was kind of a great option that we had. Not just the oral component as an option, but that oral with the concrete or the visual. Some of them actually went to the Mathies digital tools. Maybe the last leg of what we should do is just think about the math thinking. Like, what names do we want to give based on the solutions? We definitely have repeated addition. Potentially multiplication.

[Linda Macris:] Multiplication based on repeated addition.

[Connie Quadrini:] Right. And that link. Repeated subtraction and how that might also link to division.

[Linda Macris:] I think this naming of concepts and labelling things makes a very big difference in the learning that goes on in the classrooms. What she did, she repeatedly iterated. And we've used that word in math class. That means repeat. We've iterated the 2/5 again and again and again. It's not just something that is here and gone. It's something that we have a name for, and there's a reason for it. And something that may have been considered unimportant by students before because it wasn't identified suddenly is foundationally important to everything that they're doing.

NON-VERBAL REASONING - OBSERVING STUDENTS' GESTURES AND ACTIONS WITH THE TOOLS

[Speaker:] As educators it is important to pay attention to students' gestures and their actions with the tools in order to better understand their non-verbal reasoning. As you observe these students what can you learn about their understanding of the mathematics beyond what they say?

[Speaker:] So what's happening here? What did you two prepare so far? This one is 36 and this one is 57. Okay, and how do I know it's 57? Can you show me how I know it's 57? Because this is 10, 20, 30, 40, 50, 57. Yeah. And this is 57, okay. And how about this one here? It's 36. And that's the 36? And can you show me or prove to me that that is 36? This is 10, 20, 30, 36. So how much greater is 57 than 36? So I see something between the rods, something that's the same. Do you notice anything similar or the same between some of the rods that you created here? Oh okay, yep, yeah, leave it there, leave it there. Yeah, you can put it back. So you're right, it is the same. Those 2 are the same. Is there any other ones that are the same to each other? Okay, those 2? Okay, good. Oh, okay, so now what do we have here? What do we have left here that we haven't matched up yet? What do we have?

[Student:] The black one and the green one.

[Speaker:] We have the black and the green that we did match up. Can we do anything to see if we can find something that might be the same? I'll leave you and see if you can think about something, okay?

NON-VERBAL GESTURES IN MATH CLASS

[Narrator:] Gesturing, or communicating ideas through the use of hands, has recently been of interest to several educational researchers.

[Nora Newcombe:] A gesture is an action but a gesture is also an abstraction away from an action. So you're not actually lifting, and pulling, and touching, and interacting. So a gesture is a good tool for thought because of that abstract nature. On a continuum from just doing, and very abstract symbols, and words, and so forth, a gesture is really a bridge between those two.

[Narrator:] Researchers Arzarello and Edwards explain that while speech is linear in nature, gesture is more global and synthetic. One gesture, for example, can express an idea as a whole, or convey complex meanings. Gesturing plays an integral role in teaching and learning mathematics. Both teacher and student use of gestures strongly correlate with student achievement. As Nora Newcombe explains with her example of equality, gestures can powerfully express spatial information embedded in the mathematic concepts.

[Nora Newcombe:] There's also a symbolic aspect because you need to remember that the equal sign isn't about okay, give me an answer now. It's about equal, and you can see that I just can't resist making a gesture to really communicate that. And I think, to me, that shows the, you know, spatiality of the concept.

[Narrator:] Students naturally use gestures as they explain their thinking.

[Student 1:] Basically I could picture like a -- that you have the ribbon, and then you divide the ribbon into five pieces. And then if you had those five chunks of ribbon, each decoration would take two out of two chunks. So then if I have one decoration take two chunks of ribbon, another decoration take another two chunks of ribbon, so it means two decorations so far and four pieces of ribbon and then there'd be one piece left over and that would be fourths. You can only fit half a decoration on that because you don't have two to fit.

[Narrator:] Through their research, Goldin-Meadow and Wagner discovered that gestures can reveal thoughts not expressed in speech, and can also change those thoughts. Gestures actually free up the learner's cognitive resources, allowing them to invest more effort into their tasks.

[Student 2:] I was thinking 2/5. Because one is equivalent to 4/10.
40 centimetres of a metre. So 40 plus 40 is 80. Plus 80 plus 40 is 120. Plus 40 is 160.

[Student 3:] 200, 240, 280, that's 7 because we can't go up to 8, it would be 304 because we only have 3 metres, we can't go over.

[Narrator:] Research described in the Paying Attention to Spatial Reasoning document highlights how gestures may be incredibly powerful in helping to form pathways in the brain, and in developing conceptual understanding. It is therefore beneficial for teachers to augment their verbal instruction of spatial concepts with gestures. Students can be encouraged to use gestures to more effectively communicate spatial information that may be too complex to express in words. Observation of these gestures can also give educators valuable insight into students' mathematical thinking.

SPATIALIZING - OVERALL DESCRIPTION

[Narrator:] As highlighted in the Paying Attention to Spatial Reasoning document, strength in the spatial ability is related to success in math. Yet, spatial reasoning is not a separate content area or strand of mathematics. Instead it is a process that can support learning and communicating across the strands. As educators, we need to consider how we can spatialize our math programs by identifying the visual and spatial aspects within the content. These are often revealed through key words and phrases in the math curriculum. For example, many curriculum expectations include the use of concrete materials for students of all ages, so they can investigate relationships and uncover fundamental concepts, especially before generalizing to formulas or procedures. The phrase, "Using a variety of tools and strategies," highlights the importance of creating multiple representations, for example, of numbers or measurement concepts and then making connections among them. The verbs within the expectations, like "create," "build," "compose," "decompose," "perform," "represent," and, "model," reveal the actions that promote and develop spatial reasoning as students investigate mathematical concepts. The phrase, "Using a variety of mental strategies," encourages students to engage in spatial visualization activities such as composing and decomposing objects and numbers in their minds. Students are also building powerful mental images as they create a variety of visual representations to display data and relationships. By making a shift and paying greater attention to the spatial aspects in the curriculum, educators can offer students opportunities to develop their spatial reasoning. Research also indicates that teachers' own comfort level with spatial reasoning is related to students' growth and spatial skills throughout the school year. Some educators who are personally uncomfortable with spatial reasoning may avoid including spatial activities in the classroom. Educators can help their students by increasing their own familiarity with teaching and learning spatial thinking skills, keeping in mind that spatial reasoning is malleable and can be improved at any age.

SPATIALIZING - PRIMARY - SPATIALIZING THE CONSOLIDATION

[Teacher:] The problem, how much greater is 57 than 36, was designed to encourage the use of concrete materials. As part of their assessment for learning while observing during the problem solving, teachers identified and took note of spatial look-fors. Like watching for the use of ten rods and paying attention to gestures and actions with the tools. Teachers decide to explicitly highlight the spatial aspects that surfaced in the problem solving as they plan for the consolidation.

[Connie Quadrini:] We need to be strategic with who we're picking and why. We're looking strategically at which students are looking at the negative space, which is through comparison. Which ones are adding on, like adding up. And how are they adding up? Like some might add up right from that number. Some might go to the next ten and then add. So if we want the 3 different ways to come up, adding up, comparing, and the subtraction, then that's what the core teachers are looking for. Hopefully the naming of that math will come out in the action, but if they haven't used the terms, then we can bring that back. Oh, so what you're referring to is adding up. Can you explain, Toby, when what you did? And Miss Quadrini is going to show it on the board.

[Student:] So we showed 36, so we put three 10s and the 6, which is the green one, dark green. And then we made 57 with four yellow rods, 1 orange rod, 1 blue rod, 1 white rod, another blue rod, another white rod and the black rod. And then we put 57 subtract 36 equals 21 and that was our answer.

[Teacher:] Okay, can you -- they did something really important, I think. Can you tell Miss Quadrini what you did with your pencil? Explain that.

[Student 1:] We put our pencil to divide where it ended so that we knew that 9 minus 6 equals 3 plus the other white one and then we added on from there.

[Teacher:] They put their pencil here to show that these parts were the same, but this is the part that was the difference. And that's where the 9 minus 6 came from, and that's the 3 if you count the white units, 1, 2, 3. That's where it was different. And then Clara said they added on. So Clara, what were the numbers that you added on to figure out how much greater 57 is?

[Student:] We added 3 plus 1 plus 9 plus 1 plus 7.

[Teacher:] And what did that equal altogether? What was the total?

[Student:] 21.

[Teacher:] 21. So Clara and Tyson added on to figure out that 21 was their answer.

[Teacher:] Can you explain to me what you did?

[Student:] First of all we started off with the number 57 made by rods.

[Speaker:] Okay, and what rods did you use to make the number 57?

[Student:] 5, 5 orange rods and 1 --

[Student:] Black rod.

[Student:] We subtracted 3 orange rods.

[Teacher:] So you subtracted 3 orange rods. And what number did that equal?

[Student:] That equaled 27, but then we took away the 7 and put -- and added a 1 because 7 minus 6 equals 1. 57 minus 36 equals 21.

[Teacher:] So they used subtraction.

There was one more.

[Student:] Then we got 5 orange rods and underneath it we put 3 orange rods.

[Teacher:] Okay, then what did you do?

[Student:] Then we got a black rod and 7 rods.

[Teacher:] So 7.

[Student:] And then underneath it we put a 6. And then we got the 7. Put it on top of the 6. Then we put a white rod in between because 7 minus 6 equals 1. Because 50 minus 30 equals 20, and that's how we got our answer 21.

[Teacher:] So you took the matching tens away. So what I think what Victoria and her group were doing were comparing the numbers. They were comparing the numbers. When they put one number on top of another number to match them all up.

[Teacher:] We want you to kind of think of some of the big ideas we came up with today after doing the two activities that we did.

[Teacher:] So what did you learn about comparing two-digit numbers? Okay, so think first in your head.

[Student:] I learned today that you can use different strategies ^M00:05:36 by counting or matching or subtracting.

[Student:] I learned that comparing two-digit numbers is like adding and subtracting or adding on numbers.

[Student:] So I think we learned how to add and subtract with the rods, right?

[Teacher:] What things stood out today?

[Student:] I learned that you can add and subtract rods to create new ones, and you could also trade them to get like a 10 rod or 2, 5, like 2 yellow rods.

[Teacher:] That was really good. And did you feel that was important? Did that help you today?

[Student:] Yeah, it did.

[Teacher:] Okay, great. What else did you learn today?

[Student:] We found interesting that you can add them together to make a bigger rod.

[Teacher:] So is that one strategy that you can say that we learned, would be adding?

[Student:] Yeah.

[Teacher:] Adding the numbers together?

[Teacher:] What was another strategy that we learned today? Clara?

[Student:] We found out that we can add and subtract in the same question to get to the right answer.

[Teacher:] So you can add and subtract, so subtracting was another strategy?

[Student:] Yeah, because we added and we subtracted, and then we got to our answer.

[Teacher:] And there was one more type of strategy that you used.

[Student:] Comparing the rods.

[Teacher:] Comparing the rods, good.

SPATIALIZING - PRIMARY - SPATIALIZING THE CURRICULUM

[Narrator:] As educators work to spatialize the grades 2 and 3 expectations involving operations with whole numbers, they identified the spatial aspects and the wording of the curriculum expectations. Phrases that stood out were, "represent," "compare," "compose," "decompose," "use a variety of tools," "mental strategies," and, "concrete materials." Recognizing the emphasis on concrete materials, they decided to focus on whole number rods since students had recently used them in their linear measurement unit. They carried out some introductory lessons that involved composing and decomposing numbers with the rods.

[Connie Quadrini:] In the last lesson, we had introduced the relational rods on day one and kind of looked at combinations that made 10. That was kind of the focus of the first lesson. And thinking back now to our last lesson, we were focussed in on having students represent numbers that were greater than 10.

[Narrator:] After teachers plan the problem for the lesson, they solved it using concrete materials to anticipate student thinking. They developed their math learning goal and the math process focus for their lesson by interpreting the expectations with a spatial lens. Based on suggestions from the Paying Attention to Spatial Reasoning document, they strategically highlighted ways to bring attention to the spatial reasoning.

[Connie Quadrini:] One of the things that, you know, we're kind of bringing to the forefront in the lesson is this focus on spatial reasoning and how it can support students in becoming much more flexible with composing and decomposing number within the context of operations, addition and subtraction. On the right side we're adding in explicit references to how we are supporting students in developing their spatial reasoning.

SPATIALIZING - PRIMARY - THE MINDS ON

[Narrator:] In the "Minds On," teachers had students engage in a visualization strategy to encourage the creation and use of mental images.

[Teacher:] Today we're going to do another fun activity using the rods, but this time we're not going to be touching them just yet. Remember how we used this staircase?

[Students:] Yeah.

[Teacher:] Okay and we used it to help us remember certain things about the rods. I want you to have that image that picture in your head, and what I want you to do is I want you to close your eyes. So now that your eyes are closed, remember visualizing means make a picture in your head. So I want you to make a picture in your head of a line that is 28 units long.

[Narrator:] After the visualized pair-share, teachers led a class discussion to reveal the variety of solutions.

[Teacher:] Let's start with this one right here because it's right in front of me. Okay, so Nicholas, do you want to share what you did?

[Nicholas:] 10 plus 10 equals 20.

[Teacher:] Right.

[Nicholas:] And then plus 8 equals 20.

[Teacher:] How did you represent the 8, because nobody could--not everybody can see what you've made, so can you tell us how you represented the 8?

[Nicholas:] I thought 8 ones could equal--equals 8.

[Teacher:] Okay.

[Student:] So we put 2 orange rods which makes 20, and then we put a brown rod which makes 8.

[Student:] We used 2 orange rods which would equal 20, and then we used 4 red rods which would equal 8.

[Student:] We used 2 yellow, 5 rods; one 10 rod and one 8 rod.

[Teacher:] Is there anything you notice about the four different representations that we came up with?

[Student:] They're all the same size.

[Teacher:] And when you say size what do you mean? Can you give me another word for size? Who wants to help? Yes?

[Student:] Length.

[Teacher:] Okay. Is that what you meant?

[Student:] Yeah.

[Teacher:] Perfect. Matthew? What did you think?

[Matthew:] They all have an orange rod.

[Teacher:] They all have an orange rod, and what was the value of the orange rod? Justin?

[Justin:] Ten.

[Teacher:] Is there anything else that you notice? Alexander?

[Alexander:] The rod--everybody had different theories of how to make what they need out of the rods, and they used different size and amounts.

[Teacher:] You're right, everyone came up with a different representation using different rods to make the number 28.

[Student:] We put them--the first two make 10 and the last one makes 8; that makes--

that equals up to the number 28.

[Teacher:] Yes, you're right, you're right. So that leads them to make a number sentence too that we can come up with.

[Narrator:] At the end of the "Minds On," teachers made connections between the concrete and numerical representations. This was intended to support students in the problem solving as they created number sentences to match their actions and concrete representations.

SPATIALIZING - PRIMARY - REFLECTION AND NEXT STEPS

[Narrator:] After the lesson, teachers reflected on their experience with spatializing number concepts.

[Teacher:] For me, that was an important moment, recognizing, oh my gosh, maybe some of these kids, it's too difficult for them to recognize that these numbers, 10, 9, 8, are made up of units. And I took that for granted myself when I was teaching the Grade 3 students two and three digit addition and subtraction. And then seeing these and the benefit that it could have, helping them to reach the same solution or the -- achieve the same expectation.

[Teacher:] I like how the rods, especially the rods, we can use them. Last year we used them linear. This year we're using number sense. This is easy for us for fractions. Like I can see it going into different places and I like that. I like that it's so flexible of a tool to use. And if, now that, even these grade ones have experience in the primary division, it's something that's consistent and common and I think that will help them in every grade just building on their knowledge.

[Narrator:] Teachers also planned next steps to further develop student learning and to transition to other visual tools.

[Connie Quadrini:] Some of the things we might be thinking about next-step-wise is that idea of the linear and the line. So, you know, we might think about having students, then, you know, coming back to the line to be able to anchor what they did because then that's getting to the number line and maybe potentially even open number line for some of this work. And some will still need the rods for a longer period of time, but that's OK. That experience in developing flexibility and being able to visualize quantity, compose and decompose numbers, was really equipping them for the real work in the classroom, which is being able to recognize the mathematical thinking, be able to see the assets, what students can actually do. What do they know? What do they understand? And then what are they still struggling with? And then what could they do to move that thinking forward? That is the real work of the classroom teacher when it comes to mathematics.

SPATIALIZING - INTERMEDIATE - SPATIALIZING CURRICULUM EXPECTATIONS BEFORE, DURING AND AFTER

[Speaker:] By focussing in on the spatial aspects in the curriculum expectations teachers supported students in gaining conceptual understanding of operations with fractions.

[Connie Quadrini:] There's also the add and subtract fractions with simple and unlike denominators. And again it's referencing using a variety of tools. And it does also talk about the algorithms and, you know, maybe part of what we'll think about is how we can in -- kind of make -- help kids make that link between the use of the tools and the representation of the tools and what that means for the algorithm.

[Linda Macris:] The algorithm, so often people think, 'oh well give them the algorithm and then work with it.' But I've found that if the students work with the fractions long enough then what they'll do is they will develop the algorithms on their own. So, you know, we'll put up anchor chart and, you know, how to add and subtract fractions and they're the ones who will tell me. It's a really powerful experience. And they can base all of that on the evidence from their learning.

[Connie Quadrini:] Well and the other thing I'm hearing you talk about is that idea of that conceptual understanding that feeds to that development of the algorithm and it's those memorable experiences that help it go into long term memory.

[Linda Macris:] And if you can make it a positive experience for them the more you can join an experience and a mem -- a feeling the longer they'll remember it too.

[Student:] So we know that one decoration needs 2/5 of a metre so we showed that. And we're just going up by 2/5 each time until we're getting -- well we're getting mixed numbers here. We'll be going all the way up to 7 decorations; and we know from our pictures before that we're going to have 7 and a half decorations.

[Student:] So we're just repeatedly adding 2/5.

[Student:] Yeah.

[Student:] And at the end we'd have 1/5 left so it'd be half a decoration.

[Student:] Yeah.

[Student:] So that's 7.5.

[Student:] 7.5. So 7 and a half decorations.

[Linda Macris:] They certainly had the idea of a unit and a repeated unit. So that iteration of the 2/5 and adding it in again and again and again was clear in a number of the solutions that we saw so it was that-- that idea this common unit. Even yesterday I would -- I was thinking more about adding but then we had those students who were using the compliments and subtracting. I went, 'oh well maybe that will happen with our group as well because that's really just -- an opportunity to talk about multiple different operations.

[Student:] 5/5 plus 5/5 equals 10/5. Plus another -- plus 10/5, plus 5/5 equals 15/5. But then we can't have 15/5 because that would be 7 and a half decorations. So you would have to subtract 1/5 and then that would equal 14/5.

[Linda Macris:] And then we did have the one student who took the total number of fifths there were in the 3 metres and then subtracted out the 1/5 so we had the addition, we had the subtraction, but there also that was touching on the idea of a compliment. So you had the 14/5 but you also had the 1/5, which we didn't get a chance to go into but.

[Connie Quadrini:] You can go back to the same task and the same annotation to pull out that idea of the addition that compliment. Then we've got this idea of the repeated addition of fractions, and then how we can, you know, kind of, link that to the multiplication of the fraction by the whole number. We even got the multiplication. Like how great was that.

[Linda Macris:] The student who did that was just going, okay 2/5 plus 2/5 plus 2/5. Okay I know what this means.

[Student:] For the first one I did 7 times 2/5 to get 14/5 because I, kind of, just saw that as the shorter way to do it because I, kind of, just multiplied the fraction because I saw that there were 7 decorations but like, and it was made up of 2/5 so then I multiplied that by 7.

[Linda Macris:] Okay so instead of repeated addition.

[Student:] Yeah.

[Linda Macris:] You did multiplication.

[Connie Quadrini:] In grade 7 students are going to divide whole numbers by simple fractions so that's one of the operations they're going to be introduced to. And the emphasis is using concrete materials, which is really neat. It's going to work nicely exploring some of these tools.

[Linda Macris:] It could definitely have gone into division as well.

[Connie Quadrini:] Because I --

[Linda Macris:] It was there.

[Connie Quadrini:] Yeah it was there.

[Linda Macris:] It was just in the examples we chose to use.

[Connie Quadrini:] Oh and I really think this one. So what's really neat is this idea of two and a half decorations per metre.

[Linda Macris:] Which is division.

[Connie Quadrini:] Yeah when-- Yeah when you think about the 3.

[Linda Macris:] Yes.

[Connie Quadrini:] Right because they've taken 3. They've partitioned into fifths. If we take 3 metres and we divide it by 2/5, so it's that idea of saying how many 2/5 are there in 3 metres? Well it's really revealed through their representation. So 2 and a half per metre, which is really that idea of a reciprocal of 2/5, which is 5 halves. Although many of the kids were actually iterating the 2/5 up to 7 times but the concept is ready for talk of division. How would that help you when you had to figure out how many decorations? Like this is something really important you're talking about that there are 15 1/5 in 3 metres and you talked about how many were in 1 metre so how will that thinking that you've done help you determine how many decorations there are in the 3 metres? How many decorations you can make within 3 metres.

[Student:] Well because we know that each -- did they say that each decoration takes up to 2/5.

[Connie Quadrini:] Mm-hmm. That's right.

[Student:] So if we know that then we could divide 15 by 2, 15 1/5 by 2.

[Connie Quadrini:] Hey maybe you could show some of that. That'd be awesome.

[Student:]15/5 divided by 2.

[Connie Quadrini:] Hello. You had a question?

[Connie Quadrini:] And then of course on the grade 8 end in terms of all 4 of the operations they're now starting to, kind of, link all of those together through problem solving, right. And you know that task we've been thinking about might be a great one to connect that.

[Narrator:] All operations can be uncovered through the ribbon task. During subsequent lessons teachers can strategically revisit previous student solutions to highlight examples of various operations as well as make connections between these operations.

SPATIALIZING - INTERMEDIATE - SPATIALIZING THE MINDS ON

[Linda Macris:] Today's task - we're going to start off with Minds On and I need you to visualize this. That means that, if you want to, close your eyes. Think about it. To quickly sketch out what you think this problem might look like. And here's the problem. So close your eyes. You have three metres of ribbon. Each decoration that you are going to be making with the ribbon is going to need 2/5 of a metre. How many decorations can you make? You have three metres of a ribbon. Each decoration needs 2/5 of a metre. How many decorations can you make? When you think you've got an idea open your eyes and then turn and talk with the person beside you. Your elbow partner.

[Student:] Basically I could picture, like, you have the ribbon and then you divide the ribbon into five pieces. And then if you had those five chunks of ribbon each decoration would take two out of two chunks. So then if I had one decoration takes two chunks of ribbon. Another decoration take another two chunks of ribbon. So that means two decorations so far and four pieces of ribbon. And then there'd be one piece left over and that would be for - you can only fit half a decoration on them.

[Student:] I was thinking 2/5.

[Student:] Two is equivalent to four tenths, which-

[Student:] Forty centimetres of a metre.

[Student:] Yeah. So 40 plus 40 is 80, plus 80 plus 40 is 120, plus 40 is 160. 200, 240, 280. That's 7 because we can't go up to 8 it would be 300 and --

[Student:] Because we only have four metres and we can't go over.

[Student:] Yeah.

[Linda Macris:]They had an end. They had something that was in their mind. I didn't get the sense that anybody was closing their eyes and visualizing and not able to come up with anything. So they did have some entry point.

[Connie Quadrini:] So I wonder how much the practice of visualization in that first Minds On supported them in thinking about that problem very conceptually.

[Linda Macris:] What they had in their head they immediately were able to transfer into a form that other people can see. So that was something they felt very comfortable doing.

[Connie Quadrini:] So I think that's, kind of, an important thing around that whole having kids practice visualization because it's helping them create that mental image, which really reveals a lot of the mathematics.

[Linda Macris:] And they didn't do the fallback, which so often happens with the fractions, and oh, I'm just going to use the numbers and add them together. But, I don't know what that looks like. They all seemed to have some idea of what it looked like.

SCALING UP OR DOWN - OVERALL DESCRIPTION

[Narrator:] According to the Paying Attention to Spatial Reasoning document, spatial reasoning can involve scaling up or down and can be described as imagining objects or amounts as proportionately larger or smaller. This reasoning is critical to understanding many mathematical concepts including proportional reasoning. For example, using objects like manipulatives can help students visualize what scaling up or down really means when working with ratios and rates. Scaling up or down also plays an important role as students develop their number sense of quantity and operations. When performing operations, such as four times seven, tools such as the rekenrek allow students to represent their thinking and then scale down to something that is more familiar, such as two times seven. Students can then scale up using a doubling strategy to determine the answer. Tools like fraction strips can support intermediate students as they explore multiplication of fractions. Consider this problem: If one pipe cleaner is 2/5 of a metre long, how long are five pipe cleaners? As they solve the problem, they can iterate 2/5 five times which builds on the knowledge of operations of whole numbers. They can clearly see how their visual representation scales up by a factor of five and can make the connection between repeated addition and multiplication of fractions by whole numbers, as is articulated in the Grade 7 curriculum expectation. As students of all ages work with the tools, they create the visual images that allow them to scale up and down in their minds and think more proportionately about quantities and operations. In this resource, primary students scale up and down in units of ten as they solve their problems while students in the intermediate grades use scaling up and down strategies in order to solve their fraction problem. Related resources for the junior grades are included.

SCALING UP OR DOWN - PRIMARY

[Narrator:] According to the Paying Attention to Spatial Reasoning document, scaling up or down involves imagining amounts as proportionally larger or smaller. Students in the primary grades are developing this ability as they work with larger numbers and make sense of our base ten number system. For example, they can decompose a two digit number into tens and ones. When students start with a ten rod and realize that they need five of these rods to represent fifty, they are scaling up. Primary educators can further develop this skill by engaging students in visualization activities. For example, students can visualize objects or amounts as growing by two times or shrinking in half.

They can also scale up and down with concrete materials to help create those mental images.

SCALING UP OR DOWN - INTERMEDIATE

[Linda Macris:] Last year when we were working on the proportional reasoning, we started off with the caterpillar problem, which was in my mind at the time just a too simple problem for the age of students that I teach. So I'm teaching grades 7 and 8 and the number of different ways that they solved this problem was just mindblowing. So some of them used fair share and they just proportioned out the leaves. Some of them used the table of values. Some of them used skip counting. Some of them used number thinking and they just manipulated the number and they were able to come to a solution. And then on top of that they were able to use another way of thinking about it to try to confirm their thinking. When they were doing their table values, we talked about how what happens if we change the orientation a bit and structure it differently and use a ratio table. And so you can see where the original ratio is on there but how can you manipulate that? Can you double it? Can you halve it? Can you triple it? Then it moved into unit rate. Well, what if we changed it into a unit rate and then that unit rate allows us to see any one of the numbers that we might use along the way. And this goal is to find friendly numbers that they feel comfortable with and can use to find their solution. And the flexibility that that gave the students, it was, it just opened their minds so they could see how many different ways they could move this. And the path they said okay I have to fill in that column, I have to fill in this column, I have to fill in every single column, and now they could just draw out a ratio table.

[Connie Quadrini:] That ended up equating to like the scale factor. This idea of the ratio table, the scaling down to get the unit rate and then this notion of then scaling up. Sometimes it was scaled up by factor 6 and sometimes we scaled down and then scaled up by factor 12. That was really quite an important thing that came out of that caterpillar task, how we would scale up quantities and what that would look like.

[Linda Macris:] It was again this freeing up of their ability to think about a problem in many different ways and what a friendly number is to one student might not be a friendly number to another student and they used what they needed, what they were comfortable with to be able to come to that solution.

[Narrator:] It is beneficial that students can visualize what they are doing when they scale up or down numerically so that they can develop conceptual understanding. Sometimes with standard algorithms or procedures, students are not sure about what they have calculated.

[Student:] Underlined in green it's already two fifths. One fifth that's another decoration.

[Student:] Ok. 1 and 1/5.

[Student:] We can make a line. Then it's another decoration.

[Student]: Mhm.

[Student:] Then a line. Then another decoration.

[Student:] A line.

[Student:] This is pretty much all we gotta do. There we go. And then two more goes off. There we go.

[Student:] Ok wait. So this isn't..

[Students:] 1, 2, 3, 4, 5, 6, 7 and a half.

[Student:] Whatever we did here was not right.

[Narrator:] Although the students calculated their proportion equation correctly, they were unsure of what their answer of 6 actually represented. Creating a visual representation helped students to conceptually understand the problem.

[Student:] Those are opposites, right?

[Student:] You can make 2 decorations with 1 metre.

[Narrator:] Based on the boys' observations that two decorations can be made from one metre, the boy scaled up to find the number of decorations in three metres. They understood what their answer of 6 represented and then applied their knowledge that 2/5 of a metre was 40 centimetres to figure out what could be done with the left over ribbon.

[Student:] Six times 40 because we've got the answer was 40, so we did six times 40 and as we said it equals 240 centimetres, a blue ribbon. And, you said, a ribbon you'd need 47 metres of ribbon per decoration. So we can make one more with the leftover ribbon.

[Connie Quadrini:] And this group actually as well later they came back with this

[Linda Macris:] And they've got a ratio box here is what they're doing - if they just had that. So they're bringing in their prior knowledge from our ratio unit.

[Narrator:] Giving students the opportunity to visualize and then create representations of what happens when scaling up or down, can provide them with the conceptual understanding that underpins a numerical calculations and algorithms.

VISUALIZING - OVERALL DESCRIPTION

[Narrator:] The Paying Attention to Spatial Reasoning document explains that spatial visualization is particularly important in mathematics learning and achievement since it helps learners both understand and create mathematics. It describes visualization as a specific type of spatial thinking ^M00:00:24 that involves using our imagination to generate, retain, retrieve, and transform well-structured visual images. It is sometimes referred to as thinking with the mind's eye. As psychologist, Dr. Sue Ball explains, it is not just one skill.

[Dr. Sue Ball:] What is involved in spatial visualization? And make sure, looking at that definition, that we really recognize that there are different abilities that go into making up that spatial visualization skill. So the ability to generate, retain, retrieve, and transform well-structured visual images are all different abilities. And so a student could have -- could be able to generate but have difficulty holding onto it or remembering it. Or retrieving it when they've got to bring it back again. Or transforming, they could be able to come to an understanding of what that looks like but then have difficulty if they have to transform it, manipulate it in some way. Working memory could be a part of that process. They can hold it in mind but then to be able to manipulate it and work with it and see it in different ways is hard for them. So recognizing there's a lot of variability in what's required for spatial visualization.

[Narrator:] It is important that we, as educators, have students engage in visualization exercises to practice and develop these skills. Students also benefit from sharing their visualization strategies to reveal the many different ways problems and solutions can be imagined. In this resource, educators and students engage in visualization activities to develop their spatial visualization skills. They can also work with various manipulatives in order to create mental images that they can later retrieve and use to solve problems. There are also matching activities included that can be used at the primary, junior, and intermediate levels to help learners link visual representations to their numeric values.

VISUALIZING - PRIMARY

[Connie Quadrini:] The research suggests that spatial reasoning, in particular, spatial visualization is really important when it comes to supporting mathematics achievement. So part of that work is having the teachers themselves think about using the tools, but then having them practice what it means to visualize. I'm going to have you think first as an educator in this activity that I'm going to ask you to do, and then I'm going to ask you to think after that through the lens of your students. So close your eyes. I'd like for you to visualize a line that is 28 units long. If you use the whole number rods which rods would you use to show a length of 28 units? Talk to someone about the picture, the images you have in your mind.

[Teacher:] The two orange ones and the black.

[Teacher:] Yeah, and the brown. See I pictured just because I'm just thinking like different. I did the two orange, the five, and then I did the two red in the one frame.

[Teacher:] Yeah.

[Teacher:] So there's like a whole bunch. Then I even--in the beginning, I guess because we've been talking about it so much, I even visualized like 28 white ones like right off the bat.

[Teacher:] Yeah, that's what I did.

[Teacher:] Yeah, and then right from there then I went to the next one.

[Teacher:] Yeah, I did too.

[Connie Quadrini:] Let's hear some ideas from the group. So what was kind of the first image you had in your mind?

[Teacher:] I went like for the two orange and the brown.

[Teacher:] I had visualized the two orange, the yellow, and three white units.

[Teacher:] That's what I had said.

[Teacher:] Okay.

[Teacher:] That was my first.

[Teacher:] But then I traded in my head.

[Teacher:] Because you were thinking brown and you traded?

[Teacher:] Well, no I wasn't. I initially thought the yellow--to be quite honest I couldn't remember the colour of the eight, and that's what I was thinking the other kids would be doing in my class too, right, so then--

[Teacher:] Then they would compare; they'd have it in front of them, right.

[Teacher:] That's right, they would have it physically in front of them, but I thought you know what; they would probably go for the friendly number so they'd find the five, right, because they could actually see that that's half of the orange. So I took the five, and then what I did is I said, “you know what? They're probably just going to count by ones.” But then some of my kids, and I remember from an activity we had done the last time we got together, one of my kids started trading with the two. They’re like, “oh instead of using all the whites, the ones, we'll just use the two, the red.” So then in my head when I traded I took two of the whites put those back and I put the red rod plus the one for the--

[Connie Quadrini:] Okay, so I want to just bring something out here. So you know we started with talking about visualizing you know ten, ten, eight. Then we've talked about variations of the eight, so we've talked about five and three ones. Then we talked about sets of two, right, so that four sets of two units. So what's fascinating is even though we've got the least amount of rods with ten, ten and eight, but there are some advantages of students thinking about these different combinations. Well, why would there be an advantage to that?

[Teacher:] They're decomposing within that number.

[Connie Quadrini:] Absolutely, and what I loved is you described in your minds spatially what you were doing. That trading in, you were doing all that kind of spatial reasoning to be able to take eight and decompose it in different ways. That's what we want kids to be doing, because that spatial reasoning is going to support them again with that flexibility in number.

[Teacher:] I first visualized 28 single ones, and then I started to trade for some reason.

[Connie Quadrini:] Okay, that's quite all right. The 28 ones actually does reveal the unit, right, the single unit. But then what you're doing is you're taking ten ones and grouping them to make ten. So now you're composing rather than decomposing. The experience with the tools helps them to create the mental images, and that's what we want.

[Teacher:] What comes into your mind when I say 29?

[Student:] Two orange rods and a blue rod, that goes straight into my mind.

[Teacher:] If you were to visualize 20 and 5 what would you see in your head?

[Student:] Two yellows, one orange, and one five are the rods.

[Student:] I would see again, two orange, and one five; it's one yellow.

[Teacher:] And what would the difference between them look like?

[Student:] One orange and a yellow.

[Connie Quadrini:] The rods actually help them create mental images in their mind, and in fact, we heard many students talk about I could see that picture in my mind.

VISUALIZING - INTERMEDIATE

[Connie Quadrini:] So one of the other more recent tasks that we, as educators, engaged in was comparing fractions through the practice of visualization. So I strategically picked certain fractions and I asked educators on the collaborative inquiry team to close their eyes and to try to visualize which fraction was greater. And I had encouraged them not to refer to common denominators. For you, personally, what's been most impactful for you as an educator with some of the things we've been exploring around fractions?

[Linda Macris:] I think first was the ability to visualize and using benchmarks with the visualization. And the ease with which we were able to do that having used some of the tools like the fraction strip tool and the relational rods. And the fact that you said, "Please try to avoid using common denominators," which is what a lot of people would fall back on. But now that we've been using the tools, that I didn't even feel a need to think about common denominators because it was simply easier to visualize it.

[Teacher:] We felt that 3/7 was a little less than half and 5/9 was a little bit more than half and at that point it didn't really matter. As soon as we knew that 5/9 was a little more than half, we knew that 1 and 5/9 was the larger fraction.

[Linda Macris:] It took a shorter amount of time and it required a lot less load on my brain for figuring that out because otherwise I would have had to hold onto the two fractions and then I would have had to figure out the multiples in my head. And then I would have had to apply that to the numerator and denominator. And then, you know, can I compare that way? And the number of ways I could have made errors using that method in my head were far greater than just being able to visualize it.

[Connie Quadrini:] What was really fascinating is that as we heard different team members talk about how they were determining which fraction was greater, they were drawing on the complement in order to compare fractions.

[Teacher:] We looked at the complement. Again, we immediately put the ones aside because we knew they were irrelevant but there's 4 pieces missing from the 3/7 and there's 4 pieces missing from the 5/9. And the 7th slices are larger than the 9th slices so then 1 and 5/9 must be greater.

[Connie Quadrini:] They were using ideas like benchmarks. They were looking at quantity and then be able to make decisions around the fractions that were represented.

[Teacher:] I thought, "OK. Let's see how far away the 7 is from the midpoint of the 9ths. So I took my 9 and I cut it in half. So the halfway point would be 4.5 and the 4.5 compared to the 7 -- OK. So that's a little bit of a movement, 2.5 points from the middle point. So let's do that with the 5th. So I took my 5th and I cut it in half and you get 2.5. But that's only a half point to get to the 3. So it's less of a space so 7/9 has to be bigger.

[Teacher:] That's how I thought of it. How far it is from the centre towards the whole. So if you have the 7/9 and you take the centre of that, what the spacing is between the half and the 7.

[Teacher:] Quantity.

[Teacher:] And then the same with the 5ths. So we start half way through the 5ths and how far that is towards the whole. It's a lot smaller space.

[Teacher:] So that's a benchmark approach.

[Janine Franklin:] The idea of exploring fractions either through a set model or an area model or a linear model, actually constructing a picture in our head has helped us start to understand the relationships between numbers in a new way.

[Connie Quadrini:] Now they're not thinking of common denominator as their first go-to.

[Linda Macris:] I thought that was really important and important for thinking about how do you now make sure that the students are using that visualization.

[Teacher:] I want you to visualize, in your head, 5/6.

[Student:] When I hear a 6th like the fraction 6, I automatically think of a hexagon.
^M00:04:39:15
I don't know why, but, so I visualize like 5/6 of a hexagon shaded in and there's the 1/6.

[Teacher:] Why a hexagon?

[Student:] Because it's easy split into a 6th.

[Teacher:] And what would be 1/6?

[Student:] Well, we used the patterning tools today during our math class and we had to like relate 1/6 of a hexagon in a small green triangle. That's why I thought right away as soon as you said 1/6. I was thinking, "Oh, it's a hexagon, 5 green triangles equals 5/6 of a hexagon."

[Teacher:] What do you visualize, then, for 7/6?

[Student:] I can visualize -- we have a pink double hexagon. So it's like a double. And then what happens is you can put the 6 triangles there. So you go 1 whole and then you add one little triangle more to add a 6th.

[Teacher:] So that's the visual in your head.

[Student:] Yeah that's the visual in my head.

[Student:] I still see the hexagon but I don't see the pink double. I just see one yellow, like a full one and then like another 6th of the yellow hexagon.

[Narrator:] It is important that students develop flexibility in their thinking when visualizing tools they have used. For example, we can challenge students to visualize a different whole by asking questions like, "What does the green triangle represent if the red trapezoid is the whole?" With the new whole the green triangle now represents 1/3 rather than 1/6 when the yellow hexagon was the whole.

[Connie Quadrini:] They're drawing on so many strategies that actually anchor to visualization.

VISUALIZING - INTERMEDIATE - A MATCHING ACTIVITY TO SUPPORT VISUALIZATION

[Connie Quadrini:] Starting at the beginning of this year the teachers transitioned into thinking about fractions in terms of part-whole relationships and we explored several different models. A task that I had them do is I had actually represented several fractions with the fraction strips tool, and I also had kind of a matching numeric representation that connected to each of the visual representations. And I had--I had distributed them and I had asked team members of the collaborative inquiry to find their match, and the amount of mathematics content knowledge that emerged from that task was incredible.

[Linda Macris:] In order to match up you have to have a basis for comparison, and I found that if you had the numeric representation of the fraction that it was a lot easier to visualize what that fraction looked like using benchmarks. I know that this 7/8 is pretty close to one whole, so immediately the idea is you go around and you start looking for somebody who has something which is close to one whole.

[Sandra Fraser:] We hadn't really talked about the compliment for fractions, and I know that the one that I had was the complement of what Linda happened to have. But it wasn't the way we would typically look at the fractions.

[Linda Macris:] So many of us think in terms of the fraction that we have as opposed to the piece that isn't part of the whole; that complement.

[Sandra Fraser:] So it was the importance of looking, you know visualizing the fraction and saying well, what about looking it from the opposite like not knowing--I'm not even sure we--I was able to give it a label at that point in time, but I knew that there was something. So I said to her, well I'm going to follow you around until you find something that's better, but right now I'm just seeing that this is the one that matches what I have the closest. We were iterating the different pieces or where it--whether it was the complement of the fraction.

[Linda Macris:] And then once you've got that then you can refine your thinking. Okay, so if it's 7/8 the denominator means that it's a smaller piece of the whole and that the other person who might have 5/6 it's a large--that missing part is a larger piece of the whole and then you can compare, because visually you can see what part is missing from this picture and how close is it to that person who has the image and match it up that way.

[Connie Quadrini:] They were further partitioning to find equivalency and then matching that up to an appropriate numeric representation.

[Sandra Fraser:] And the equi-partitioning like it was all the vocabulary that started coming out when we debriefed it after, but we didn't actually necessarily have the vocabulary attached to it as we were going around and trying to find our partners.

[Connie Quadrini:] And it I think brought to the forefront for the educators in the collaborative inquiry that this is the work that how important spatial visualization is in particular to the work they were doing in fractions, but more broadly around what that means even in other content areas.

[Sandra Fraser:] It was a beautifully put together task; it's all about the mathematics and deepening our own understandings.

PROPORTIONAL REASNONING - PRIMARY - DEVELOPING PROPORTIONAL REASONING THROUGH UNITIZING

[Speaker:] According to the paying attention to Proportional Reasoning document, the ability to unitize is one of the most obvious differences between students who reason well with proportions, and those who do not. When people unitize, they can envision each rectangle as being both one rectangle, and 4 squares simultaneously. And in total, there are 5 units each with 4 squares. The rectangle example illustrates how spatial reasoning can help students develop the ability to unitize. Which will in turn, support their ability to think in multiplicative terms.

[Connie Quadrini:] Through several experiences now that the students have had with a number of lessons that we've done, they're starting to be able to recognize units, but that are made up of one units.

[Student:] This is 8s. And another one is 8.

[Student:] 8?

[Student:] 60 is the --

[Student:] It's 5.

[Student:] 5? This is 4.

[Student:] Wait. That's 3. This is 4.

[Student:] Okay.

[Connie Quadrini:] It's difficult for students to think, well this was actually 3, but now it's actually 1. Right? So 1 rod can actually represent 3. That's a very abstract concept for students. But it's through the concrete tools, they can see immediately that that is made up of 3, 1 units. So that unitizing is really supporting them. And one of the beautiful things that happened -- breakthroughs I would say for students. Is they were able to see that the orange rod was a very special rod.

[Student:] These are 10 rods. So this is a 10 and this is a 10, so it would equal 20. These are worth 2. So that's 20, 22, 24, 26, and 28. And this one equals -- 10 and 10 is equal to 20 rods. Then 8 of these equal 28 rods.

[Speaker:] Is there anything else that you notice?

[Student:] At the beginning, all of them equal 10.

[Student:] They all equal 10 at the beginning.

[Speaker:] Are there any other combinations that you can notice between the different value of rods that we used? Yep?

[Student:] Like the middle ones -- all the middle ones equal 10?

[Connie Quadrini:] Okay. So we've just done the minds on. That's great. Some pretty interesting things that came out. What do you think?

[Speaker:] I loved how they saw the 10, the 10, and the 8. I loved how that -- and then you captured it so well by boxing it out --

[Speaker:] Yeah.

[Speaker:] And really having those kids that didn't see it. It really showcased what they were talking about.

[Speaker:] This was good, because this showed them that there were other ways to make 10 as well, right? Yeah.

[Connie Quadrini:] And that's what's really important about the role that unitizing can play in the context that we've been looking at it. Representing number as well as operations with number.

PROPORTIONAL REASONING - USING SPATIAL REASONING TO SUPPORT PROPORTIONAL REASONING

[Connie Quadrini:] So last year Roselawn decided that their content focus would be proportional reasoning because they had noticed that students were struggling in that area. We look specifically at ratio and rate within proportional reasoning. What's important about it, you know, two quantities that are changing simultaneously. We began looking at tools, concrete tools, manipulatives, that could be used to not only represent mathematical thinking, but I often say reveal important mathematical structure, important mathematical concepts within representation.

[Sandra Fraser:] It's how they organize even when they're using like two colour tiles to be able to represent, you know, the pattern within that context. It's very different from making them into piles as opposed in to arrays which helps them see the mathematics, that there is an actual mathematical pattern that is happening.

[Linda Macris:] Well, if they represent the ratio in an array, then it structures it in a way so that you can see the groups, and then you can see how many groups there are,
^E00:01:22 ^B00:01:23:05
and then that gives you the what are you multiplying by. So it takes you from the additive way of thinking to a multiplicative way of thinking.

[Narrator:] Next, students can transition from the array to the table of values. The array reveals how the quantities are consistently increasing, not only individually but in relation to each other.

[Linda Macris:] And then they use that to create ratio boxes, and they were able to explain why the ratio box worked and how one number became another, what they did to that, so it was, you know, oh, I'm multiplying by this many groups and I can show it in my array, and so that the one idea confirms the other. And the connection with the ratio table is that they can take any two little columns from that ratio table and put them together and go, "Well, that's a ratio box which works for this." ^M00:02:19:15
In the past, you could go, "Okay, well, I think I multiplied by five, but I'm not sure why I figured that out." Here you can actually show it both in the array and in the ratio table where those numbers came from.

[Connie Quadrini:] That ended up equating to like the scale factor and how we would scale up quantities and what that would look like in the context of an array, being able to see some of the important ideas within proportional reasoning.

PROPORTIONAL REASONING AND THE ARRAY - USING THE ARRAY IN PROBABILITY

[Connie Quadrini:] Starting at the beginning of this year, the teachers were looking at the Grade 6 EQAO assessment from last year. I actually took one of those questions. It's a probability question. And I have the teachers solve that as the -- as part of the first session when we came together. I asked them to solve it in two different ways, but one of the ways had to involve using a tool -- a concrete tool. Some of the deepest thinking around the mathematics, around part-to-whole relationship as it relates to probability emerged from the use of the tools.

[Mark Warling:] I tried to model one of the Grade 6 EQAO questions. This is an unusual solution; I'll try to recreate it. So Justin has a bag of 40 coloured tiles. Without looking, he reaches in and pulls one tile out. Complete the table below to determine the probability of choosing a red, green, blue, or yellow tile. So there's a couple of things given. There's 6 red tiles. So why don't I get out my 6 red tiles. Because that's a given. And they also gave him 10 green tiles. So we'll get those out and see what we can manipulate with those.

[Connie Quadrini:] And don't forget the probability of choosing yellow was 2/10.

[Mark Warling:] So there -- yeah, one other -- one final given, 2/10 yellow. So 2/10, you know, that would be like a 1/5 of a whole, or 20%. But let's come back to that. I'm going to work with these first.
And I'm going to try to make an array of things and see it works. So there -- you were given 10 tiles. There's a total of 40. So let's see if we did something like that. And we have 6 -- and we have 6 reds. So we'd need a space to make 40. If that's 10 across, if I wanted to make an array -- like a rectangular array of different tiles, I would need 4 coming down that way and filling out the whole thing, that would be an array of 40.

[Connie Quadrini:] Mm-hm.

[Mark Warling:] So I have yellow, and they said 20% or 2/10. Let's see, if I made my array going this way -- let's see, 2/10. 20% of 40 is 8 tiles. There's 8.

[Connie Quadrini:] So how'd you know that 20% of 40, or 2/10 or 40 is 8?

[Mark Warling:] So 2/10 would be 1/5 of the whole.

[Connie Quadrini:] Mm-hm.

[Mark Warling:] And 1/5 of 40 would be 8.

[Connie Quadrini:] Okay, that's pretty significant, how you organized it like that.

[Mark Warling:] So if I group those together, that represents the .2 of the whole.

[Connie Quadrini:] So that, 2/10 of the whole.

[Mark Warling:] That's the 2/10 of the whole. So what if I kept my 6? These are all the tiles that I'm working with right now. And let's put it like this. Just matching up those fifths. In other words, the .2 of the whole. And I have these two. And then I'm going to fill in that -- I think we've got it now. So given 6 red tiles. Given 10 green tiles. There is my chunk that I determined of 0.2 of the whole, or 1/5 of the whole.

[Connie Quadrini:] Mm-hm.

[Mark Warling:] Or 8 tiles. I have one section here to make 1/5. So let's see, the blue tiles. So that means the remaining pieces have to be blue tiles. And I'll keep it in fifths. 8, 16, 24, 32. There's your array.

[Connie Quadrini:] So you can really see your 5 1/5 in this the way you've grouped it. And I'm very intrigued by this idea of the -- this representing 2/10. But again, how your spatial organization actually can reveal -- can support you in knowing the probability of the rest. How would you determine, based on how you've structured this, what the probability would be for the blue tiles, the green tiles, and the red tiles.

[Mark Warling:] Well what if we did that? That would represent .1 of the whole.

[Connie Quadrini:] Okay, or 1/10.

[Mark Warling:] Or 1/10 -- one 1/10, or two 1/10.

[Connie Quadrini:] Yeah.

[Mark Warling:] three 1/10, four 1/10, five 1/10, six 1/10. Seven 1/10, eight 1/10, nine 1/10 or ten 1/10 meaning one whole. But we've got this little wrench right here. Little curve ball. So what if we [laughter] -- what if we did this, and that would separate out those two. Now we've got a different piece to work with, but it still represents the whole. Now we had 1/10. We've divided it in half. Would that now be 1/20th of the whole? One 1/20, two 1/20, three 1/20, four 1/20. So that would represent your yellow. One, two, three, four, five 20ths --
For the probability of green. Three 20ths for the probability of red. And one, two, three, four, five, six, seven, eight 20ths for the probability of blue.

[Connie Quadrini:] That's cool.

[Mark Warling:] As a fraction.

[Connie Quadrini:] That's cool. It's interesting how you went to 20ths, and how you were then able to count the unit out because you were able to see those equal groups.

[Mark Warling:] The well-designed little curve ball right there.

[Connie Quadrini:] Yeah.

[Mark Warling:] Forcing me to do that.

[Connie Quadrini:] Mathematical structure. The units that happen to be represented in terms of decimal, the probability. But how they were able to unitize and see the units within their representation, and how they used those units in order to be able to solve the problem.

PROPORTIONAL REASONING - INTERMEDIATE - PROPORTIONAL REASONING AND THE RIBBON TASK

[Linda Macris:] And then it was interesting, the one group, where he took out sticky notes and started adding sticky notes and sticked them right onto the computer screen to show what was going on in his head.

[Student:] Tenths. So this is 3/10. Each of these strips is a tenth. So there's 3 tenths strips. And then since 2/5 is equal to four times this would be one decoration or 2/5 slash 4/10. That's another 4/10. And I just represented it with a sticky note. That's one decoration. That's two decorations, that's three, went up to seven. If we did one more. I could do, and then we would go over - a decoration, it'd be two fifths and that's over three. So we can't do eight, it'd be seven.

[Linda Macris:] Can you show me where the 1/5 are?

[Student:] Yes.

[Connie Quadrini:] I think a fascinating dimension to that was that whole idea of the metres and centimetres relationship. So you really named it, well, a kind of proportional reasoning that they were using, but how they were able to use the tool to make the connection. Yeah. Wasn't that neat? Because they took tenths and their tenths represented 10 centimetres.

[Linda Macris:] 10 centimetres.

[Connie Quadrini:] So the tool actually allowed for proportional reasoning. And I wondered how much in relation to the measurement.

[Linda Macris:] I wonder how much having just finished a proportional reasoning, you know, with rate and ratio, how much that influenced it, because I was surprised at the number of students who went that route. I -- as I had said before, it wasn't something that I would have thought would come so naturally and yet we saw it in multiple, used multiple times.

[Connie Quadrini:] Even the pair who used the three number lines. Like, that whole idea of being able to spatially represent those quantities and track those quantities.

[Student:] This number line is going to wholes and this one is going to fifths and this is --

[Student:] How many decorations?

[Connie Quadrini:] So what I'm noticing is this is like you've got -- you're holding onto two quantities at the same time.

[Student:] Yeah.

[Connie Quadrini:] You're holding onto the parts of the ribbon --

[Student:] Yeah.

[Connie Quadrini:] In terms of fifths.

[Student:] Yeah.

[Connie Quadrini:] And then you're tracking the decorations as it relates --

[Student:] Yeah.

[Connie Quadrini:] To the quantity used of fifths.

COMPARING OBJECTS - OVERALL DESCRIPTION

[Narrator:] We typically think of comparing objects as analyzing shapes and figures in the geometry and spatial sense strand. The spatial ability is also at work when students use concrete materials to develop their number sense. As stated in the "Paying Attention to Spatial Reasoning" document, students need experiences with concrete and visual representations so they understand that numbers are quantities that have magnitude and occupy space. Manipulatives that are linear in nature like whole number rods, allow students to represent numbers and compare their quantities by looking at their lengths. Teachers can help students to transition from whole number rods to number lines to investigate a variety of numbers and their spatial relationships. By paying attention to each end point of the number line, students can see how much space each number occupies and make comparisons based on their various relationships. As the number line is stretched or compressed students can explore how spatial relationships of numbers change depending on the end points. Students can use a spatial reasoning by representing other numbers like fractions as length and then making visual comparisons. They can also use familiar benchmarks to compare. Double and triple number lines are also excellent for holding on to various quantities so comparisons can be made. In this resource, primary teachers discuss how they used whole number rods for measurement, and we then see how their students transitioned to addition and subtraction by comparing the lengths of the whole number rods. Intermediate students made comparisons with various tools perform operations with fractions. Ideas are also given for comparing objects to develop number sense in the junior grades.

COMPARING OBJECTS - PRIMARY - ANTICIPATING STUDENT THINKING

[Connie Quadrini:] We kind of did a comparing idea, too. We should show them our equations, should we?

[Rosanna Cristello:] Sure. What I did is I used four 10s and then two fives to make the 50, and then I used another five to make 55, and used -- I used two ones to make 57, and then the second one was 36, so I used -- we used three 10s.

[Connie Quadrini:] And then it was five and one?

[Rosanna Cristello:] And it was a five and one. And then what I did was, because we're trying to find how much more 57 is, I just started taking those that were -- that matched. So the first two 10s are together, so it's the same in both numbers. Then they both have -- so the next two match, and then so do the next two, and then there's a five in 36 and there's a five in 57. And there's a one in 36 and there's a one in 57. And what's left is how much it -- how much greater 57 is, which is 21. There's a 10, two fives, and a one.

[Connie Quadrini:] We were playing with the idea of what the number sentence might look like, and so we kind of created the two number sentences, but then we were saying, "How could we show the actions of that whole idea of the matching?" So we used this tool or this function.

[Speaker:] Crossed it out?

[Rosanna Cristello:] Yeah, we just crossed them out. That's the same in both numbers. And what's left over is what's different, or how much --

[Speaker:] And then you just added those ones?

[Rosanna Cristello:] Then we just added those ones together, yeah.

[Connie Quadrini:] So we're not actually removing it from the equation, but we're highlighting it so that we can actually see what matches up, and then what's revealed, and then I said, "Rosanna, look."

[Rosanna Cristello:] Oh. So 36 plus the 21 is actually 57, but you could actually see it by -- what matches up is 36 and then what remains is 21. I can see some kids and they have the same rods.

[Connie Quadrini:] So, I'm sorry, tell me more about --

[Rosanna Cristello:] Meaning the one that we did, they both have 36, so they might say they have the same rods.

[Connie Quadrini:] Okay, so that -- so if kids said that, what would that be called? Comparison. So, again, we'll document what they say, but then we want to introduce that language. That's the comparing, right?

[Rosanna Cristello:] So I think that's how they'll -- it'll come out.

[Connie Quadrini:] So we've spent a considerable amount of time ourselves as educators really using the tools to not only explore the potential of the tools, but actually explore deep understanding of the concepts. It's really supported them in thinking about what it really means when we're adding or when we're subtracting or we're comparing numbers. They could see the difference between a comparison strategy and a strategy that involved adding up. Were students adding on or were they decomposing numbers and reorganizing in order to be able to compare?

[Children talking over each other performing math] So that's 10 plus 10 plus 9 plus one. That's 10. Ten plus 10 plus 10 plus five plus one. That's six.

[Children talking over each other performing math] Ten and 10 plus. Ten plus 10 equals 20 and one little space there I can fit one little white block. And one little, sorry, one little white rod. And one little white rod equals one. So 10 plus 10 plus one equals 21.

[Student:] When we got five orange rods, and underneath it we put three orange rods.

[Speaker:] Okay, then what did you do?

[Student:] Then we got a black rods and the seven rods.

[Speaker:] So seven.

[Student:] And then underneath it we put a six. The seven goes with the --

[Speaker:] Five tens?

[Student:] Yeah.

[Speaker:] [inaudible], Make 57, okay.

[Student:] Then we got the six and we put it next to the three 10s.

[Speaker:] Next to the three 10s, all right.

[Student:] And then we got the seven, put it on top of the six. Then we put a white rod in between because seven minus six equals one. And then we took away the seven and the six.

[Speaker:] [Inaudible].

[Student]: Yeah, moved that one. Because 50 minus 30 equals 20, and that's how we got our answer, 21.

[Speaker:] So you took the matching 10s away.

[Student:] And that's how I got --

[Speaker:] Can anybody think of a word Victoria's group used when they put one number on top of another number to match them all up?

[Speaker:] [inaudible]?

[Student:] I forgot.

[Speaker:] Okay.

[Speaker:] So I think what Victoria and her group were doing were comparing the numbers. They were comparing the numbers to find out what the difference would be.

[Connie Quadrini:] The depth of the content knowledge actually equipped them to recognize that when they were with their students and observing them.

COMPARING OBJECTS - PRIMARY - CONNECTING LINEAR MEASUREMENT TO QUANTITY

[Teacher:] So last year I began working with the St. Agnes of Assisi Group, the primary team, and they were quite interested actually in linear measure and some of the struggles that students face around using different kind of items to measure and units to measure. One of the things that we explored was the use of relational rods or whole number rods in the context of primary to explore linear measure. And what the teachers came to discover is that they could use the rods as a bridge between nonstandard units of measure and standard units of measure. So, in fact, the nonstandard units of measure which would be all the rods that are not of one unit, the white rods, that they actually were kind of in disguise made up of one unit, so that idea of a unit that could be made up of several one units. And so they spent time exploring that. We developed several lessons with the vision to be able to move them from measuring with the rods to them being able to annotate that through a linear model. As we started to shift into this year they realized that that work actually had very strong connections to number sense, in particular, representing quantities, numbers.

[Teacher:] The girls that I was observing what they were doing, I mean, they were taking something like this and they were able to show just spatially they were counting one, two, three without using the white. So they're beginning to see that in their head.

[Teacher:] They're still looking at the unit of three, but they're able to see three one units within a three rod, which is pretty powerful, and how the tool actually allows for the development of that visualization of that quantity.

[Teacher:] And then they did the first number, the second number, and then even before they filled in, one of the girls actually with her finger just finished off 4 plus 10 plus 7 - oh, that's 21.

[Teacher:] So how fascinating that that student was (a) showing through the gesture of a length of four, then a length of ten, and then a length of seven. So, again, paying attention to that and having your eye attuned to that.

[Student:] You have--I kept putting these rods to equal that, and then I counted all of them and they equal the number.

[Teacher:] I'm still stuck on the whole linear. I just like the whole notion that they can actually see the quantity as a length.

[Teacher:] Yeah.

[Teacher:] And so they're--and they're even like they notice that this is shorter, this is longer, so when they're touching it like they can physically feel that this is shorter than that one.

[Teacher:] Lovely.

COMPARING OBJECTS - INTERMEDIATE

[Connie Quadrini:] We are going to offer kids the relational rods, so let's think through maybe, like, how might we even solve it using the rods?

[Linda Macris:] Okay, well, I think the first thing is that you have to figure out what whole you can use and part of that is going to be students figuring out the relationship between the rods. So they might try to do relationships like that and realize, okay, well, those -- those -- those don't fit in. I would like to think of them as thirds, but I have still got something left over. And then they could try adding in smaller pieces to find out what exactly does this represent. And go, okay, well, so those match up, but what does that mean? So, I have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 of the smaller pieces so this would be -- each one would be one-tenth. So I have ten-tenths in total. And if I am looking at this problem and I know that I need one-fifth or two-fifths, then I would start saying, well, can I match up any other pieces and make relationships there. So if I take this piece, I know that now that that is the same as two one-tenths.

[Connie Quadrini:] Mm-hmm.

[Linda Macris:] And if I take more of them -- and because I know that it matches up here, I know that I will be able to create a relationship between the red and the orange as well.

[Connie Quadrini:] Nice. Yep. From the relational rods perspective, they were thinking about the units that they needed in order to create the relationship of fifths, compared to their whole.

[Student 1:] So this is our whole, okay? So that's one. That's two. There's three. There's four.

[Student 2:] Why are you doing it with four though? Why are you doing it in fourths?

[Student 1:] Because, we should be getting like two-fifths.

[Student 3:] Okay. Give me the greens.

[Student 4:] The greens? Thanks.

[Student 3:] Oh, do two-fifths. Get up. Oh, I worked with these before. This is a fifth, I'm pretty sure. Oh, wait. Oh, 1, 2, 3, 4,

[Student 4:] Yeah.

[Student 3:] Yeah.

[Student 4:] Yeah, that squares.

[Student 3:] Are you sure? No, dude, you can fit another one. No?

[Student 4:] Oh, here's another bar. The orange. You think? There. Okay.

[Student 3:] So --

[Student 4:] Two -- we have two -- two-fifths.

[Connie Quadrini:] Each of the ways in which different, like, each pair approached it actually allowed for quite a bit of that visualization to emerge. With the fraction strips, they were able to kind of -- they were able to unitize two one-fifths using that equivalency bar.

[Student 5:] Each of these is a metre of tape - a metre of ribbon and this is -- what you need to make one decoration. Each metre, you make believe two decorations because you don't have enough ribbon.

[Student 6:] Yeah, it's the same thing as four.

[Student 5:] That's two-fifths and that's two-fifths.

[Student 6:] Those are all two-fifths, right?

[Student 5:] Yeah. You can make two decorations with one metre.

[Connie Quadrini:] It's neat that they were able to see that they need two one-fifth units and that, again, when they dragged that out, they could see that in relation to the whole.

[Linda Macris:] It allows students to manipulate fraction strips and compare them so they can start seeing the relationships between the fractions at that point, and they can visualize it at the same time.