Active Learning: Learning to see like a mathematician

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Some claims to be considered

- What you see is central to how you reason, problem solve;
- We learn to see (and change what we see)
- Novices need to learn to see like experts see;
- 3-D visual processing is distinct from 2-D visual processing;
- Kinematic experiences are linked to 3-D visual processing - and to mathematics
- I wish the same options for my students.

Private face / Public face

- I do my mathematics visually,

  and kinesthetically, in private.
- Choosing when and how I use spatial reasoning has:

  changed the questions I pose;

  changed the methods I use;

  changed the answers I give;

  changed my communication and my teaching.

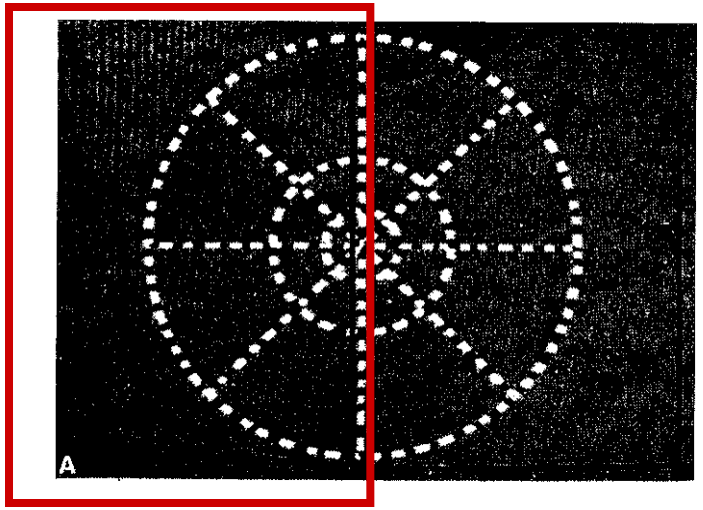
Mathematics with Eye and Hand:

- Mathematics is done in the brain
- eye is part of the brain, made external.
- eye and hand are linked in the brain:
  eye-hand coordination;

  planning in the brain;

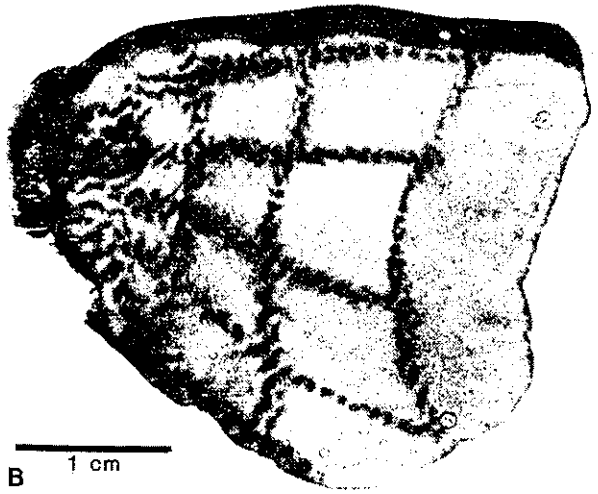
  strongly linked to emotion;
- which tools and activities build mathematics?
- links to visual, spatial and kinesthetic
- Specific experiences in 3-D.

A macaque monkey was trained to stare at this pattern.



Injected radioactive sugar to highlight blood flow. ‘Sacrificed’ while staring at the image

This is an x-ray picture of half the visual cortex at the back of its head



External and internal images share these processes

A mathematician sees … with many parts of the brain

- Work with a brain developed before mathematics existed - ‘conceptual metaphors’ (Lakoff & Nunez).
- No simple pattern of steps.
- Typically not using language parts of brain.
- Heavy use of visual / eye-hand parts

$7 \times 5 = ?$

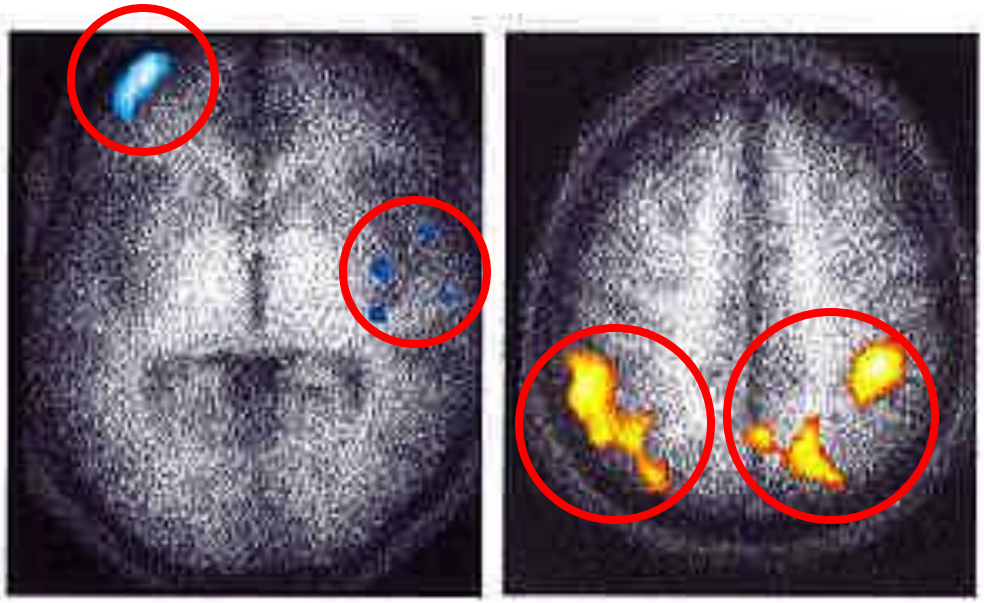
Language (multiplication table)

$7 \times 5 > 26$?

Visual (analog number line)

A mathematician works … with many parts of the brain

58+70 = ?



58+70 =

? 320

Language (tables)

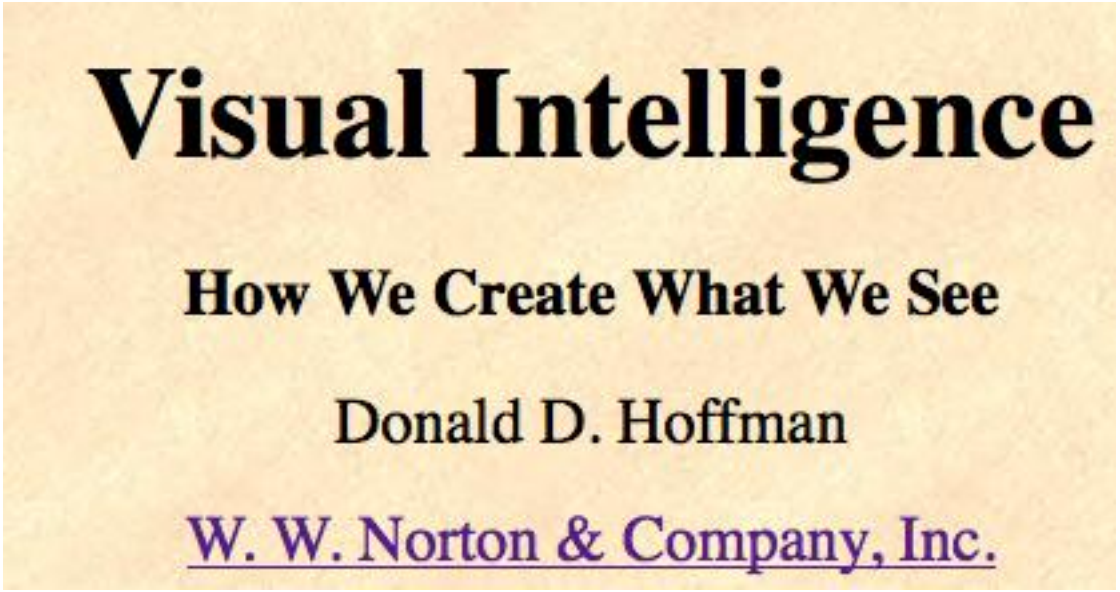
Visual (analog number line)

- Multiple independent pathways,
- Many surprises,
- Many visual steps

Do you see what I see? No!

We process, select, and construct what we see.

We create what we see



Motion 1

Motion 2

We create what we see

We can change what we see

- Students don’t see what we see!
- In mathematics we create what we see:

  from experience: eyes and hands

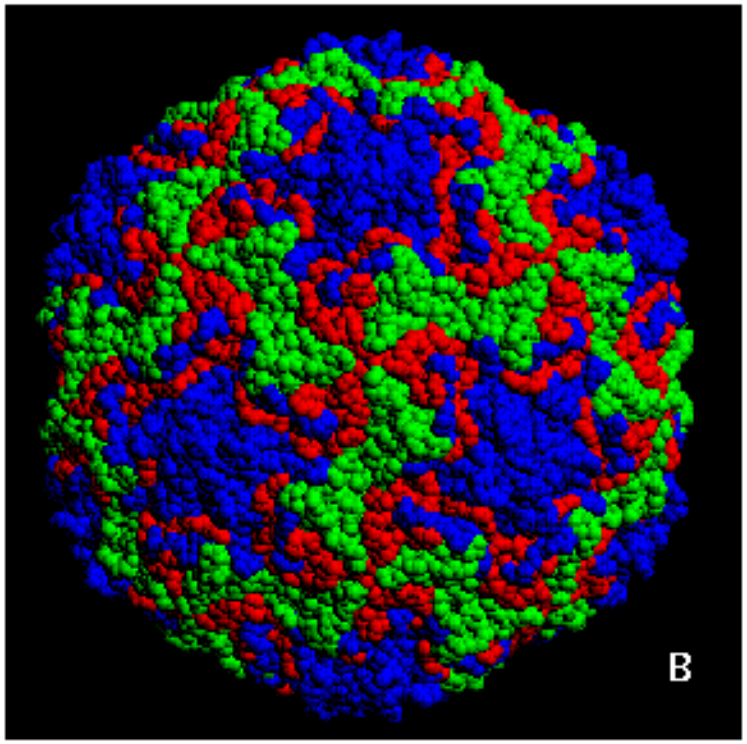
  from practice and apprenticeship;

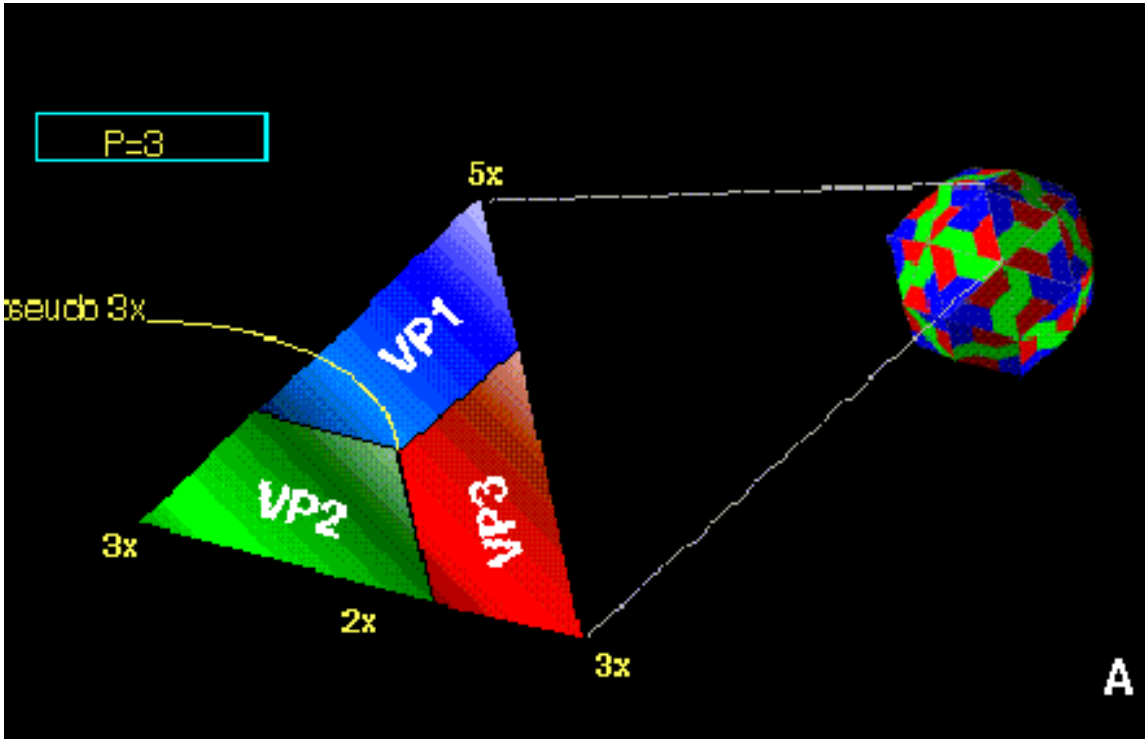
  from insight and understanding.

If I teach to you see like a mathematician, then mathematics will be easy

Symmetries in 3-Space

- Polyhedra - virus abstract images
- core concept of physics, chemistry, geometry, ...



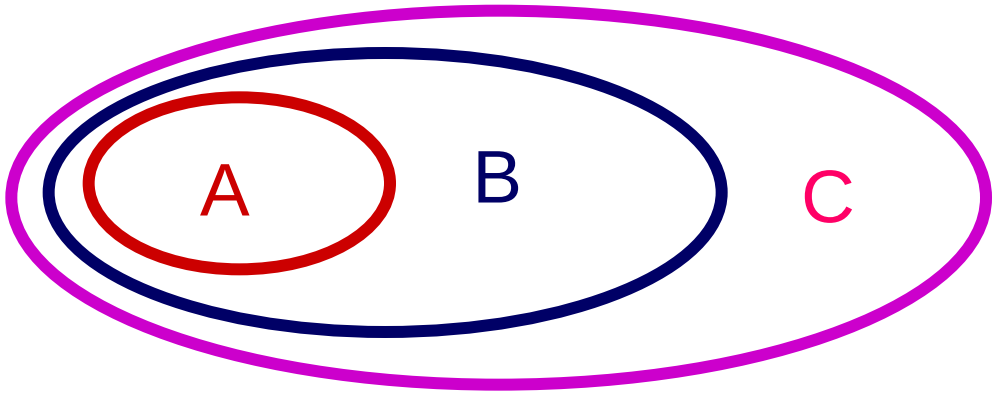


Metaphors and Transfer

Consider:

- All A are B
- All B are C
- Therefore All A are C

Is this language based?



No

Vinod Goel, Lakoff and Nunez

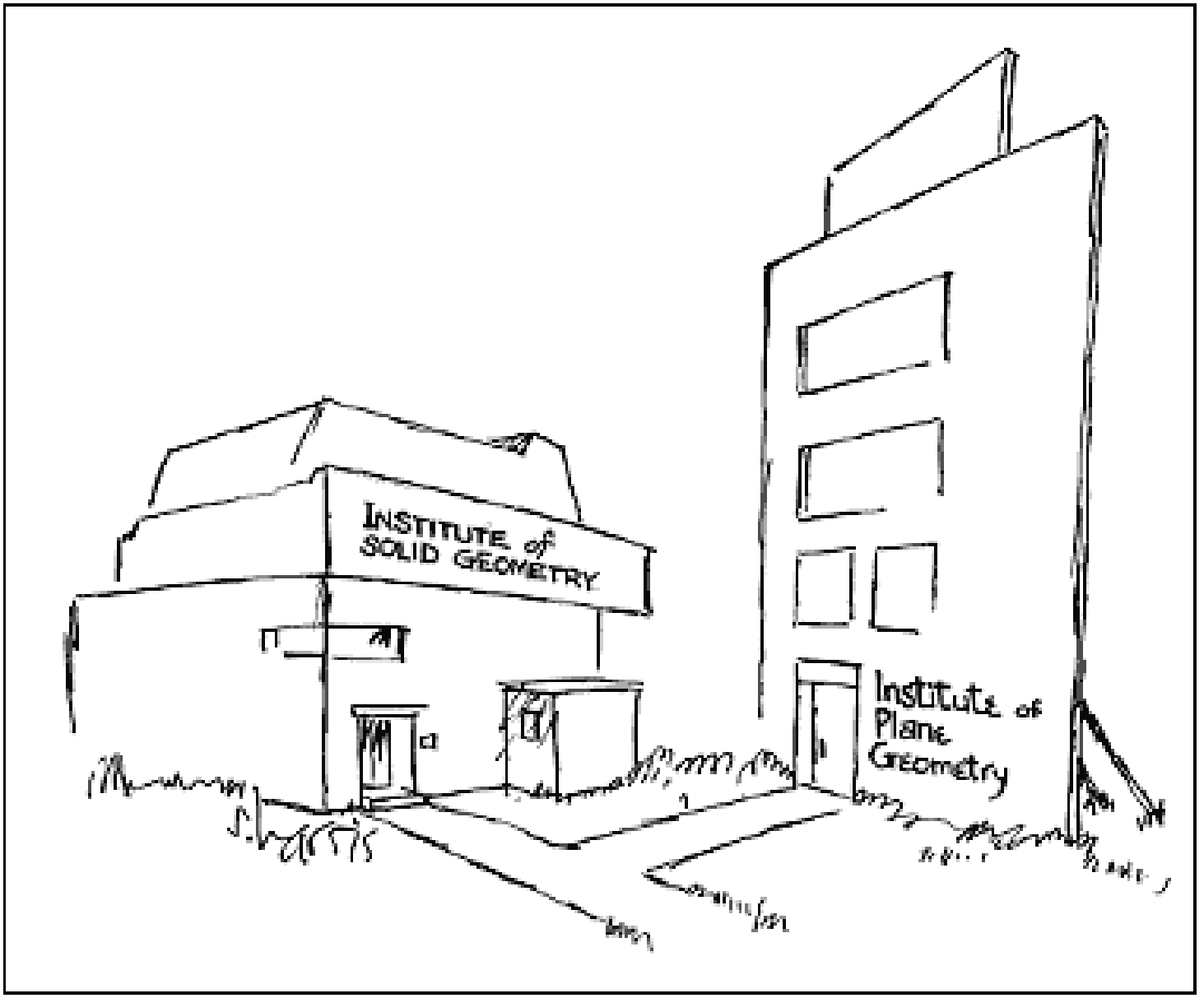
Spatial Reasoning is Essential

- Visual/Spatial reasoning is central to how you reason, problem solve;
- We learn to see (and change what we see)
- 3-D spatial reasoning is central – don't flatten the spatial child into the plane!
- Spatial reasoning is essential to doing mathematics
- Necessary for some students to access math
- Strengthens even the best students in math (and science).

Spatial Reasoning is a Gap at University Entrance

- Documented issue for first year engineers
- Weak spatial reasoning, without catch-up,
- predicts poor performance and low retention to second year
- “Engage” engineering web site
- impact on first year science, including calculus
- Is identified as source of anxiety by pre-service and in-service teachers
- Is malleable: can be developed at all ages.

3D - not 2-D



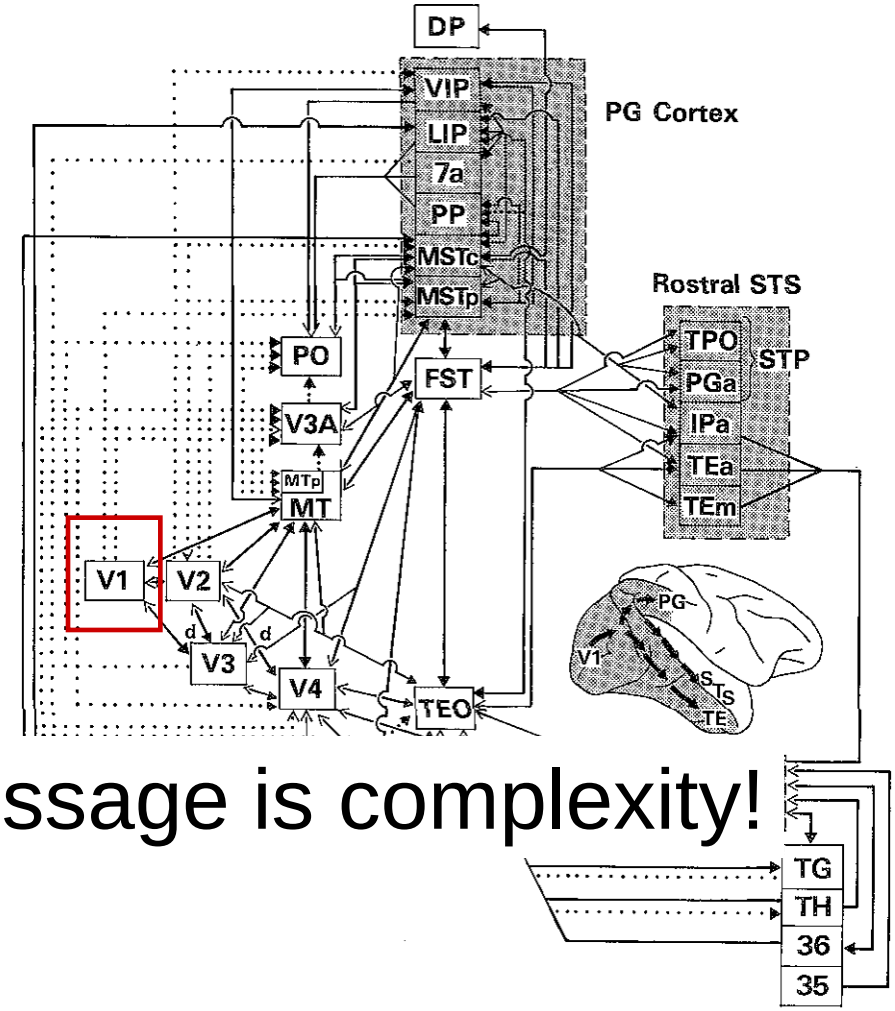
We need a mix of approaches

- We remember about 10% of what we read;
- We remember about 30% of what we hear;
- We remember about 80% of what we see and do. Jerome Bruner
- We will forget 80% of what we learn today.

A mathematician sees … with a complex brain

A ‘simplified map’ of pieces and connections.

Note: two way arrows - paths also used for images in the mind’s eye



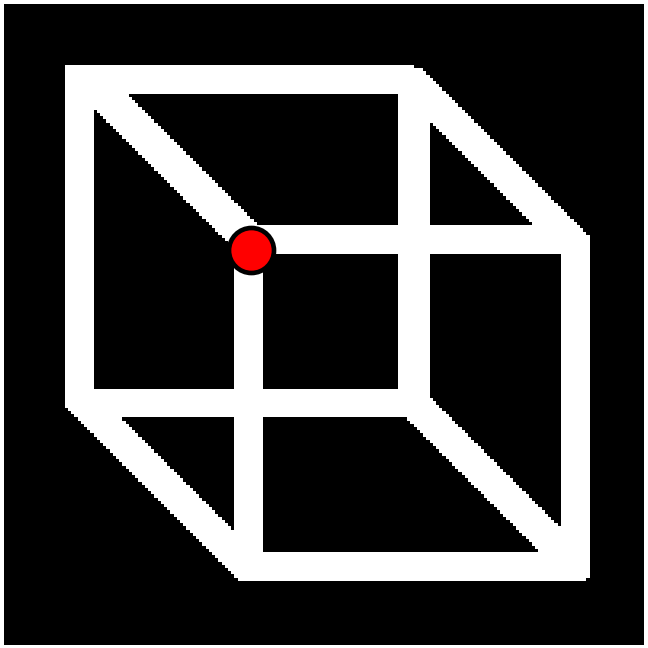
Main message is complexity!

Kosslyn

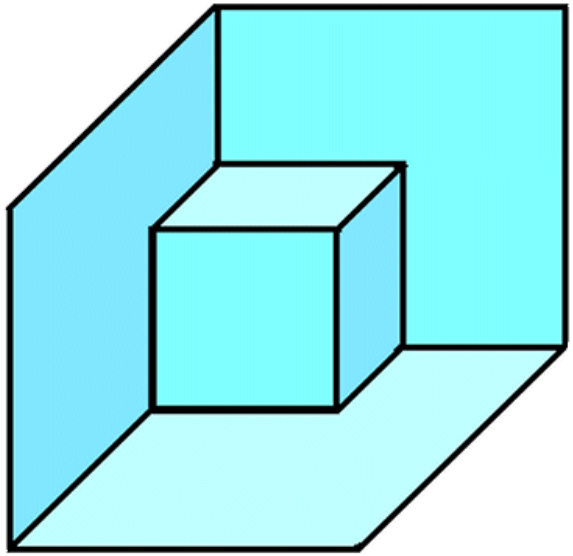
Possible Lessons from PISA

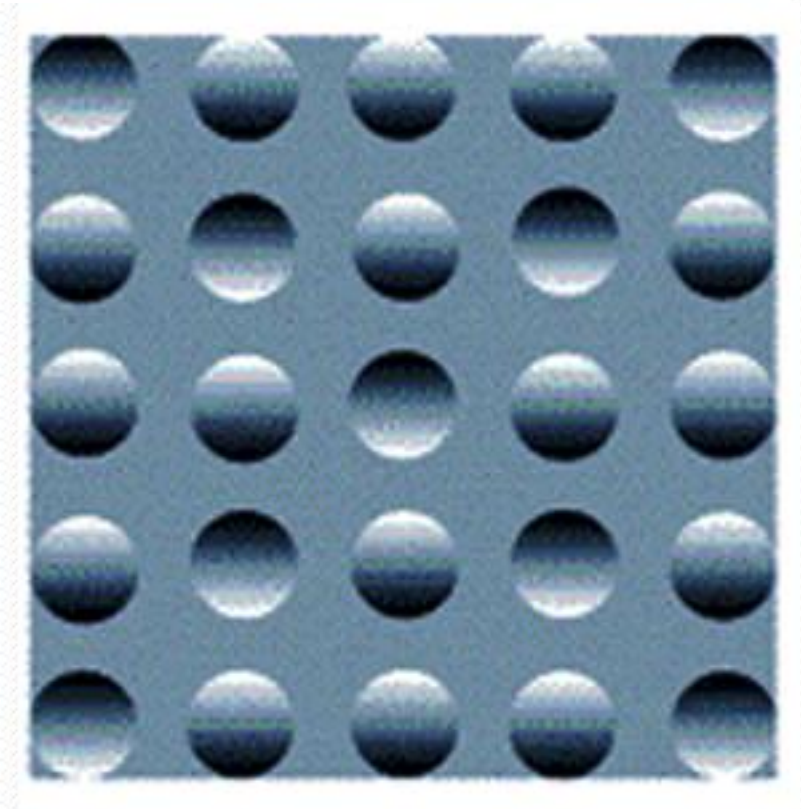
- From Shanghai
- High quality continuous professional development
- teachers of math and science in early years
- From Finland
- High quality pre-service preparation of teachers
- Treated as professionals.
- Together – they are working to improve
- Are we?

- 2-D pictures of 3-D are ambiguous.
- we may “flip” from one view to another.



Animated Cube

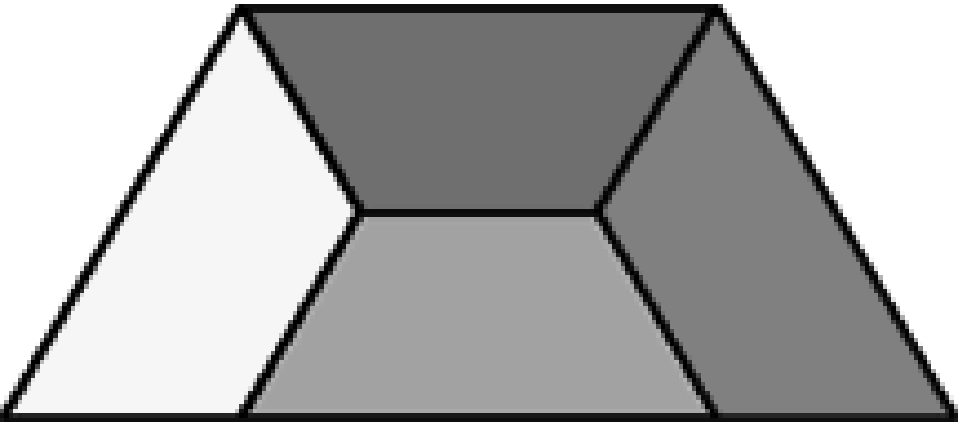




We assume that light comes from the top

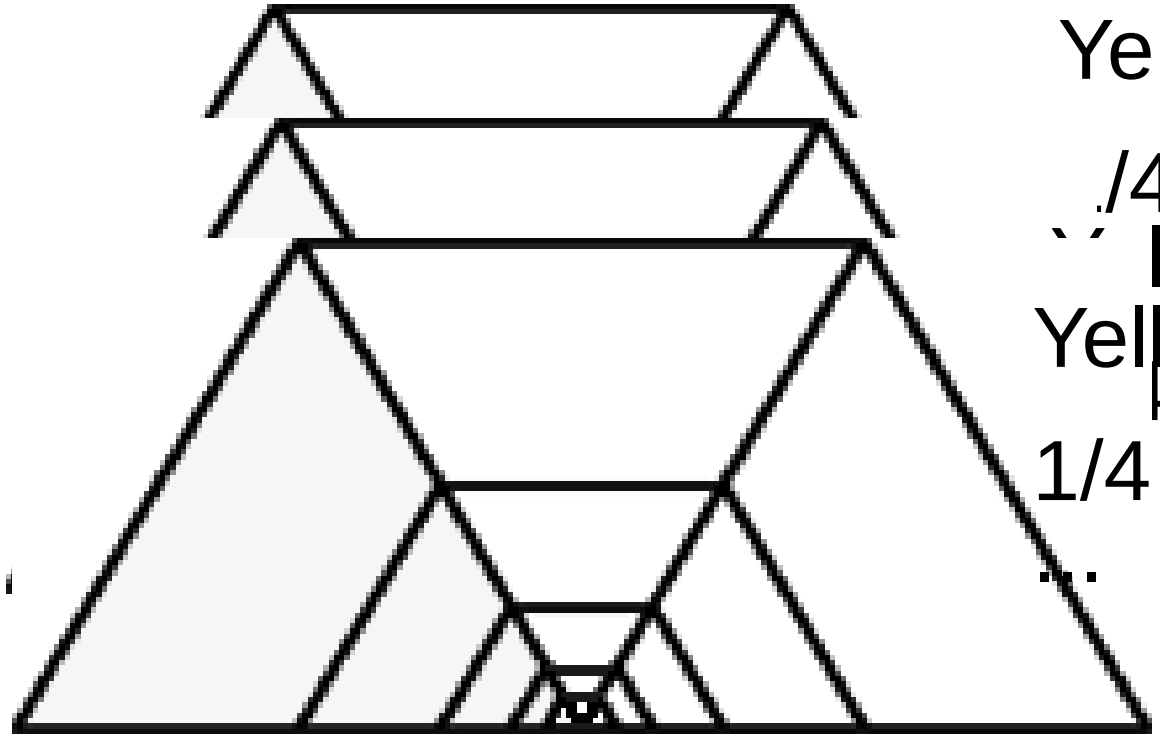
Visual Addition

A mathematician sees …



Yellow is

1/4 of whole



Yellow is

$1/4 + (1/4)^2$

Yellow is

$1/4 + (1/4)^2 + (1/4)^3$

$1/4 + (1/4)^2 + (1/4)^3 + \ldots$

$= 1/3$

Problems with the use of visuals:

- Inability to ‘see’ a diagram in different ways;
- Recognizing transformations implied in diagrams;
- Incorrect or unconventional interpretation of graphs;
- Connection between visualization and analytic thought;
- Information is determined by specific rules and conventions;
- Like algebra - needs teaching and intervening conceptual thought;

Tommy Dreyfus