Teaching and Learning Mathematics

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Since 2006 I have been involved in a variety of research projects that have focused on the development of algebraic reasoning in students from Kindergarten to Grade 9. This work has included the documentation of the relationship between the design of lesson sequences (units of instruction) and student activity, and an assessment of student learning during and following instruction. The bulk of this research has taken place in classrooms working in partnership with classroom teachers, learning coaches, principals and other researchers. The most salient thing I have learned is the capability of students to understand complex mathematical concepts when given the opportunity to construct their understanding rather than relying on rote memorization. Rote learning does not support the development of mathematical thinking, in part, because it takes away the opportunity for a student to do math. In my work I have documented Kindergarten students making accurate predictions about the number of tiles that would be required for the 10^{th} or 100^{th} position of a linear growing pattern (Beatty, 2012), and students in Grade 6 who used their understanding of patterns and graphs to design solutions for equations of the form $ax+b=cx+d$ (Beatty, 2010). Students went beyond current curriculum expectations without memorizing traditional algorithms. Instead, they explored concepts by engaging in sequenced activities using visual and numeric tools, and participating in discussions that emphasized conjecturing and justifying. In this paper I will highlights some important components of the design and orchestration of learning environments using examples from my research.

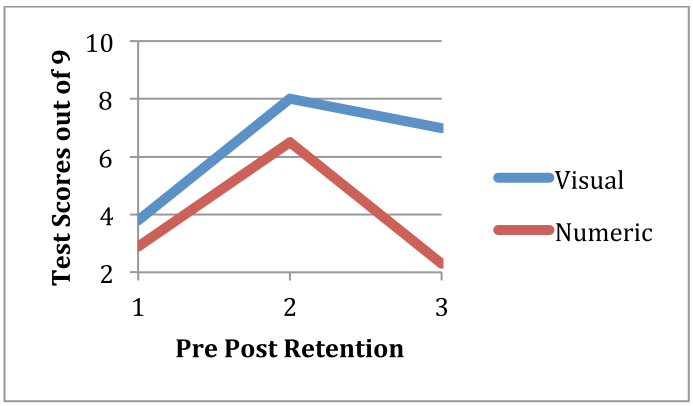
Designing Math Instruction

Designing effective mathematics instruction involves an understanding of the mathematics content as well as an understanding of how children develop mathematical thinking. In my own work, designing units of instruction has been grounded both in the research literature and in students' thinking. Each unit is made up of carefully sequenced lessons. Tasks are sequenced so that the complexity of the mathematics is incrementally increased. This means that as students move through the sequence, they are supported for success as the concepts in each lesson build onto the learning of the previous lesson. If tasks are not sequenced they may be experienced as disconnected individual tasks rather than as a series of interrelated activities designed to support the construction of mathematical concepts. Providing sequenced, interrelated activities gives students opportunities to construct mathematical understanding by bringing together their theories, experiences, and previous knowledge. When students construct an understanding of mathematics, they can then generalize what they have learned and apply their knowledge to learn new topics and solve unfamiliar problems.

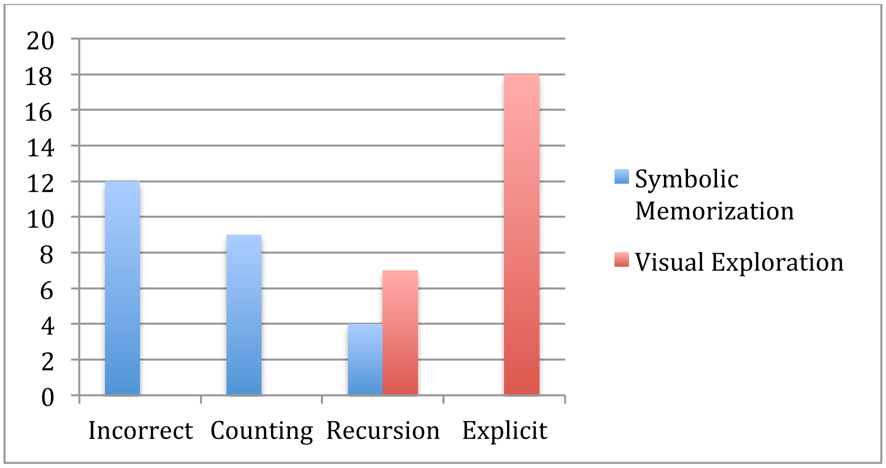
Although the sequence of the lessons is important, each lesson in itself is open-ended. The sequential nature of tasks coupled with the open-endedness of the activities allows all students, including struggling learners, access to aspects of mathematical thinking that are normally challenging.

Effective mathematics instruction includes equally prioritizing visual as well as numeric and symbolic representations of mathematics, and an emphasis on the interrelationships among different representations. This goes beyond "pictures, numbers and words." A student who understands a mathematical concept has the facility to move fluently between and among different representations, and can use one representation to test hypotheses about another. Understanding a concept presupposes the ability to recognize that concept in a variety of representations and the ability to handle the concept flexibly within different representational systems. Recent research in early algebra (e.g. Beatty & Bruce, 2012; Watson, 2010; Moss & Beatty, 2006; Lannin et al., 2006; Hoyles & Healy, 1999) has shown that students who work primarily with visual representations (including concrete patterns, diagrams, and graphs) are more successful at understanding algebraic relationships, finding generalizations, and offering justifications than students who are only taught to manipulate symbols or memorize algorithms.

As an example, in a study with Grade 4 students (Beatty & Moss, 2006) we noticed that some students used visual representations as the site for problem solving (patterns or graphs) whereas other students relied on constructing a table of ordered values and using memorized strategies to find pattern rules. We found that students who used visual representations as the site for problem solving performed better than their peers at the end of the study (post-test scores). Even more importantly, these students demonstrated an increased retention of algebraic concepts when given a retention test six months after the initial study.



In another study of Grade 9 students, those who had spent the previous year of instruction exploring visual representations of linear functions scored higher on an assessment of algebraic thinking that included multiple representations (diagrams, graphs, word problems and equations) than students who had been taught to memorize steps for manipulating symbols (Beatty, 2012). When we compared the reasoning strategy of the students, we found that students who had learned through working with patterns and graphs used more sophisticated reasoning strategies, and most were able to find and articulate the functional relationship of each item in the assessment.



In addition to developing conceptual understanding, it is equally important to provide opportunities for students to practice procedural skills in order to develop computational fluency. For example, in my work students learn about linear relationships by building linear growing patterns. However, the building of patterns (and guessing the rule for others' patterns) also strengthens students' multiplicative understanding in a way that that goes beyond memorizing multiplication facts. As students built patterns and guess the rules of their peers' patterns they need to access their multiplication facts quickly and accurately. This means that they practice their facts in a way that is less stressful and more meaningful than a timed rote drill.

In summary, effective instruction includes a slow rate of introducing a new concept, a large assortment of interrelated sequenced activities, the prioritization of visual as well as numeric representations, and multiple opportunities for practicing procedures and reviewing concepts and skills.

Orchestrating Learning - The Importance of Conjecturing and Justifying

Children learn when there is a shift from classrooms as a collection of individuals to classrooms as communities of mathematical inquiry. The focus of inquiry can come from the teacher or from the students themselves (which results in high levels of engagement). One of the most important aspects of working as a community of learners is offering conjectures, which can then be tested, refined and/or refuted. Conjecturing is a vital way for students to be able to extend their thinking beyond the specific activities they engage with (Carpenter et al., 2003). Teacher-facilitated or student-facilitated discussions are a way of encouraging students to generalize beyond the specific phenomena they are exploring. For example, students may offer a conjecture after noticing the sum of a pair of even numbers is even, the sum of a pairs of odd numbers is even, but the sum of an even and an odd number is odd. Conjecturing allows students to move beyond considering the sums of particular numbers (2+2, 3+4 etc.) to explore whether it is always the case, whether there are any instances where this does not hold true, and to begin to theorize about *why* this is the case.

Equally important is the notion of offering justifications. When students justify their solution strategies they are able to provide reasoning and evidence to validate their thinking. When students move beyond justifying their solutions based on specific examples, to justifying their answers based on more deductive reasoning, the sophistication and complexity of their mathematical thinking increases. Additionally, when students are encouraged to provide justifications for their solutions, they also develop perseverance in problem solving. For example, in a study with 50 Grade 9 students, results indicated that students who provided sophisticated justifications for their answers were also able to identify their incorrect solutions, and then worked to find alternative solutions. Students who did not provide justifications rarely identified their own mistakes, and when they did, tended to give up rather than find an alternative solution strategy (Beatty, 2012).

Technology and Teaching and Learning mathematics: CSCL and CLIPS

Computer-Supported Collaborative Learning

Part of my past research has explored incorporating collaborative technology in math instruction in order to examine what is uniquely feasible with new technologies and how these might support the shift to classrooms as communities of mathematical inquiry. Joan Moss and I researched a web-based collaborative workspace, Knowledge Forum (KF) (Scardamalia, 2004), as part of a teaching intervention in early algebra (Moss & Beatty, 2010). Two schools took part in the study. The Grade 4 and 5 classes from each school initially participated in a 12-lesson instructional sequence. Then the students were invited to use Knowledge Forum to collaboratively solve problems in order to determine whether communicating via the database would further develop their abilities to work with patterns and functions. The database was entirely student-managed with no teacher or researcher voice. The students from the two schools did not physically meet, but used the database to negotiate theories, question one another's theories, elaborate on their thinking and compare ideas. They developed a community of practice in which the offering of evidence and justification for their conjectures became the norm, and this in turn supported students' deepening conceptual understanding both of mathematical functions, and the role of justifying (a precursor to proving) in mathematical discussions.

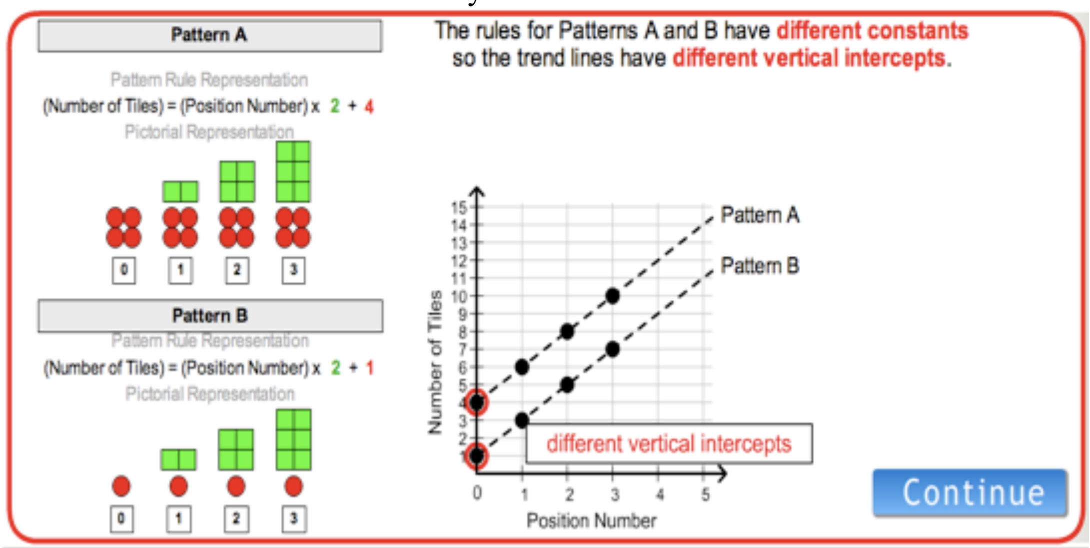
CLIPS as a Support for Students with Learning Disabilities

Online learning objects, CLIPS (Critical Learning Instructional Paths Supports) are created using flash animation and incorporate audio narration, offering students the ability to consider mathematical concepts in non-static environments. Although CLIPS was designed to be used by all learners, Catherine D. Bruce and I conducted a study in 2010 that identified two specific ways that this kind of environment supports students with learning disabilities (Beatty & Bruce, 2012).

The first support is focusing student attention. Students with learning disabilities often display attention difficulties, which may adversely affect their learning (Fuchs et al., 2008; Fuchs et al., 2007; Montague, 2007). The CLIPS computer animation was designed to direct student's attention in order that they would discern details and recognize relationships that we, as the educational designers of the activities, believe are important to discern and recognize. In each activity the aspect that we want students to notice – for example the connections between the numeric value of the constant in a pattern rule, the number of tiles in a pattern, and the vertical intercept of a trend line on a graph – becomes the focus of students' attention. The constant in the pattern rule flashes red, the red tiles that "stay the same" in the linear growing pattern flash, and the vertical intercept on the graph has a red flashing ring around it. In addition all activities have audio narration that directs students' attention to particular aspects of the task.

The second support is that mathematical connections are conveyed to the students *interactively*. Students move through a series of scenes for each activity, so that the mathematical concepts are introduced in a logical order of increasing complexity. The animation creates opportunities for students to interact with the material by providing activities in which the co-action between user and environment exists. Each representation is linked to the other representation so that as students create one, they can see the corresponding changes in the other. Students work with dynamic objects that are constructible, manipulable and interactive.

This study resulted in increased achievement in students with learning disabilities. Students constructed a deep understanding of linear relationships, and exhibited greater confidence in their mathematical understanding, as evidenced by becoming contributing members of the classroom community.



Screen capture of activity to compare pattern rules that have the multiplier but different constants. In this activity, the words "different constants" and "different vertical intercepts" flash, the red circles representing the constant part of the linear growing patterns flash, and the red rings around the vertical intercepts of the graph flash.

New Research: Culturally Responsive Education

In 2011 I began a study that looks at how we can make mathematics relevant for all students, particularly First Nations students. Current mathematical teaching practices are aligned with many aspects of Aboriginal teaching in that both emphasize experiential learning, modeling, collaborative activity and teaching for meaning over rote memorization and algorithm efficiency. An inquiry-based approach to teaching mathematics has the potential to respond to the needs of Aboriginal students through incorporating traditional cultural activities. Elder Stephen Kejick of Iskatewizaagegan First Nation (Shoal Lake #39) speaks of the importance of numeracy in a traditional Anishinaabe knowledge framework, specifically in the creation of traditional structures or items, which must be precise to demonstrate honour and represent their meaningful purpose (2011, personal communication). The precision needed to create cultural artifacts, such as Ojibway beadwork, reveal the cultural importance of mathematical thinking. However, there is a paucity of culturally relevant mathematics materials in school districts currently serving children from Anishinaabe communities.

My current research is focused on how we design and implement mathematical instruction that is culturally responsive and meaningful because it is aligned with the cultural paradigms and lived experience of students. To date we have worked with three Anishinaabe communities (two Ojibway and one Algonquin) and their respective school boards. These are the broad research questions I am currently investigating:

1. What is the impact of making connections between Anishinaabe culture and mathematics instruction on mathematical achievement of Aboriginal students the primary grades?
2. Does the introduction of meaningful cultural contexts increase the confidence and engagement of Aboriginal students?
3. How can we engage the wider community (parents, Elders) in the design and delivery of mathematics?
4. How do teachers learn from Anishinaabe pedagogical practices to achieve an equitable and inclusive mathematics classroom for Aboriginal and non-Aboriginal students?

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