Dr. Alex Lawson,
Lakehead University

What has your research revealed about the effective learning or teaching of mathematics?

My first area of research that is central to effective instruction is sadly, old but still highly relevant: teacher knowledge.

Teacher Knowledge

Gap Issue #1. If as a research community we have an agreement about the fundamental role of teachers' mathematical content and pedagogical content knowledge in children's learning then this must be part of our focus. Regardless of instructional methods deemed more or less effective none can be implemented well unless teachers understand the mathematics underpinning the instruction as well as the related pedagogical concepts. Does a gap exist between what Ontario teachers know and what they need to know in content knowledge in order to undertake the types of reform outlined in our mathematics curriculum and related documents? (*Expert Panel Report on Junior Math*, *Guides to Effective Instruction*, etc.)

Research. In 2007 Stienstra and I examined 558 pre-service students' mathematical knowledge in order to better design our methods course. We used the faculty's mandatory content exam of 15 word problems set at the Gr. 6-7 level, administered at the end of the first term, to do so. About one quarter of the student body had considerable difficulties with the exam (12% failures, 15% marginal passes with the cutscore set at 75%). Ninety of the 558 students had taken tutoring prior to the exam and all students were halfway through their 36-hour math methods course. The following year the exam was taken upon entry into the program, (in most cases prior to tutoring and, in all cases, the methods course). The rate of difficulty rose to 43% (21% failures and 22% marginal passes). These findings are not surprising as prospective teachers are themselves products of mathematical instruction, which has been implicated with poor results, at least in the United States (Battista, 1999).

Given these results, we investigated what course design would best deepen this group's understanding of mathematics to a degree that they will be capable of undertaking the main tenets of the reform. To answer this question we turned to Ball, Hill and Bass' (2005) theoretical description of what it means to know *mathematics for teaching*. They described an amalgam of mathematical content knowledge that is both *common content knowledge* and *specialized knowledge for teaching*. We used their example found in the *American Educator* (pp. 17-18) to explore our thinking. In this example, Ball et al. examined double-digit multiplication delineating the *common knowledge* aspects and the *specialized knowledge for teaching* teachers must have to teach this concept well. We briefly summarize their longer discussion in Table 1.

Table 1: Double Digit Multiplication Example of Teacher Knowledge Necessary For Teaching

Common Content Knowledge	Specialized Content Knowledge of Mathematics for Teaching
-reliably use the multi-digit	1) understand the meaning behind the steps in the algorithm e.g. why one

algorithm	'slides' the second partial product over one column 2) use appropriate representations such as an area model of (20 + 5) x (30 + 5) 3) diagnosing typical student errors 4) connecting the traditional algorithm to the area model 5) choosing appropriate numbers or examples (in this instance one that may or may not involve regrouping).

Initially as we examined the results we felt that our instruction needed some focus on the *common content knowledge* of mathematics. However, a closer examination of the types of errors students made lead us to believe that a sizeable group of students did not have a sufficient knowledge base to grapple with a methods course focused on learning the concepts under *specialized knowledge for teaching*. Nor would a focus on common content knowledge offer sufficient foundation. Instead, we came to realize that students needed further instruction in the fundamental knowledge underpinning these packets of *specialized knowledge for teaching*. For example, a sizeable segment of our student body could not participate fully in our class discussion and examination of 1) *the meaning behind the steps in the algorithm* without constructing a number of big ideas. First, students must understand single-digit multiplication, that is, be able to think about the number of groups and the number in the group simultaneously (Clark & Kamii, 1996) or how to *unitize*. Multiplicative reasoning is not just repeated addition as many of us were taught. Second, students must have an understanding of the distributive property (Ma, 1999) that is, that 25 x 35 can be thought of as 25 (30 + 5) or (20 + 5)(30 + 5) laying the groundwork for partial products. Third, students must have constructed the big idea of place value including recognizing the patterns that occur when multiplying by 10 (Fosnot, 2007). Continuing with the example, in order to 2) *use appropriate representations such as an area model,* students must understand the relationship between linear units and area units (Kamii & Kysh, 2006; Simon & Blume, 1994). Examining the results of our research we found that about 25% of the group had difficulty with some if not all of these fundamental big ideas of: multiplicative thinking, the distributive property, place value and the relationship of linear units to area units (necessary to the developing the first two specialized content knowledge abilities). Without more substantive support than we were already offering, these students would not be keeping up with our class discussions or attaining the expected learning.

Turning to our topic today of effective mathematics instruction:

1. Are similar numbers of teachers in Ontario unsure of these foundational concepts?
2. How would we know?
3. If this is the case what type and length of PD is necessary to adequately address this need?
4. Is it possible to make a fundamental improvement in the learning of mathematics in Ontario without squarely addressing this issue?
5. Additionally, how should we be tackling this issue within the new two-year BEd program?

Understanding and working with children's mathematical thinking.

Gap Issue 2: If we are agreed that teaching must begin with student understanding and explanation of a given concept then teachers need support understanding this thinking when it is different than their own. Furthermore, they must have knowledge of where the student strategies or underpinning mathematical principles lie on a continuum or progression and how to proceed in the development of these mathematical concepts and attendant fluency over time.

Tangential Research. I have completed data collection of a seven-year longitudinal study following fifty children in mathematics beginning in the fall of Gr. 1 with most finishing in the spring of Gr. 5. The students completed ten videotaped interviews of approximately one hour each solving mathematical problems and calculations over the course of the study. What was the progress of this group of students (forty-one percent of whom scored in the lowest decile in one or more of the *Early Development Instrument* (EDI) 'Readiness to Learn' indicators, whereas the Canadian National Sample was 27%) in reform-oriented classes?

Thus far by the end of Gr. 2 students have mostly, but not entirely, closed the gap in addition and subtraction strategies from their initial interview—in comparison to other students on the same items in other longitudinal studies of primary development. (e.g. Carpenter et al. 1998; Moser & Carpenter, 1984) as outlined in Table 2.

Table 2: Comparison of Successful Addition and Subtraction Strategies across Studies

	Gr.1 fall		Gr. 2 spring	
	Carpenter	Study	Carpenter	Study
Single digit over 10 addition	80	58	94	90
Single digit over 10 subtraction	55	50	80	80
Double digit addition	29	0 (10)[a]	74 (79)[b]	67 (80)[a]
Double digit subtraction	0	0 (8)[a]	27 (50)[b]	30 (60)[a]

[a] all methods including counting three times
[b] traditional algorithm use

At the same time I am validating Fosnot and Dolk's *Landscapes of Learning* in Addition & Subtraction and Multiplication & Division. Do these theoretical landscapes of children's progress over time in reform-oriented classes reflect these students' mathematical journeys? By Gr. 3 there are a few changes that would better capture what has happened with this group, but on the whole, the landscapes are fairly accurate and comprehensive. They might therefore, (this requires research with teachers), serve as a useful and validated continuum for teachers learning to listen to and work with children's thinking in reform-oriented classes. We are using them in our pre-service classes with good success.

Pertinent to today's conversation are some of my <u>very preliminary findings</u>, that, (as I am still analyzing), I cannot yet state definitively:

1) Without access to calculators and with a focus on mental math the students often carry out calculations in their head. They rely on their understanding of the big ideas (principles or relationships) in

mathematics to do this. For example given the calculation 403 – 107 rather than using the traditional algorithm (although some will) they may say: "403 – 100 is 303 – 3 is 300 – 4 is 296". Many of the students are capable of 2 and 3 digit addition and subtraction calculations in their head.

2) Some students are using inefficient methods of calculation that are error prone, typically doubling where they should (and sometimes could) multiply for multiplication and division.

3) Most students are willing to persevere for an extended period of time to solve a problem. They do not look to the interviewer for answers. Most of the students report enjoying mathematics.

4) The students usually make sense of the problem; and, many self-correct if the interviewer repeats an incorrect answer.

5) A sizeable group has not automatized all of their 'facts' by the beginning of Gr. 5. This lack of automaticity slows down their calculations and likely increases their cognitive load.

Overall, the cohort of students likely has a stronger understanding of number sense and much greater flexibility than would have been the case with more 'traditional' instruction of mathematics as a set of procedures. Without a control group this remains speculation. There are important instructional changes, based on these findings, which I would recommend to improve student learning and computational fluency in the elementary years to teachers using reform-oriented instructional methods.

References

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching. *American Educator*, 14-46.

Battista, M. (1999). The mathematical miseducation of America's youth. *Phi Delta Kappan*, *80*(6), 425-433.

Carpenter, T., Franke, M., Jacobs, V., Fennema, E., & Empson, S. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education*, *29*(1), 3-20.

Carpenter, T., & Moser, J. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, *15*(2), 179-202.

Kamii, C., & Clarke, F. (1997). Measurement of length: The need for a better approach to teaching. *School Science and Mathematics*, *97*(35), 116-121.

Kamii, C., & Kysh, J. (2006). The difficulty of "length x width": Is a square the unit of measurement? *Journal of Journal of Mathematical Behavior*, *25*, 105-115.

Simon, M., & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472-494.